NEW PLANE GEOMETRY

HAWKES-LUBY-TOUTON

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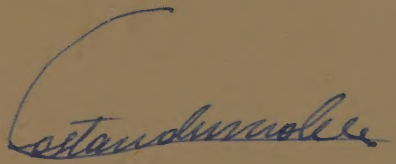

NEW PLANE GEOMETRY

Tafel Constructions
American Academy

BY

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PREFACE

The student taking up geometry for the first time faces a subject which is stranger to him than was algebra when he was fresh from the study of arithmetic. This fact has been overlooked in textbooks on geometry up to the present time. It has been the general practice to commence the study of formal demonstrations without any informal preparatory work in which the student may get his bearings. Two results have often followed: either the student floundered about for a time and then in desperation memorized the demonstrations instead of trying to understand them, or else the teacher was compelled to spend the first part of the course trying to provide an introduction which would give the student a preliminary conception of the nature of plane geometry, some idea of the reason for studying it, and a clear notion of the method of study that the subject requires. The latter situation is hard on the teacher, and it was with the idea of assisting in this important part of the work that the Introduction of the New Plane Geometry was written.

The general manner in which the subject is to be developed and the technique which must be learned are first presented to the student in this part of the book. Here he becomes familiar with the tools with which the work is to be done (the straight line, the circle, angles, triangles, the straightedge, compasses, the protractor) and learns how these tools are used in the theorems and proofs to be found later in the book.

In algebra formal proofs are rare. Hence the student who has had even a year and a half of algebra may have developed little or no ability in proving any of the algebraic theorems he has studied, such as the formula for the sum of n terms of an arithmetic series. In geometry, however, proofs of theorems are fundamental. The authors have therefore included a wealth of illustrative detail regarding theorems, hypotheses, conclusions, axioms, assumptions, and proofs, and their meanings and uses.

Geometry contains material which is of practical importance in mensuration, in elementary physical science, and in trigonometry. Geometric principles are fundamental in architectural designs and in decorative problems. Geometry also furnishes material which, properly studied, teaches the young student better than logic itself to think clearly and accurately. The authors of this book have endeavored to realize fully all the possibilities inherent in the subject. At the same time they have aimed to make the presentation accord in every feature with the best teaching and practice of current psychology and sound educational progress.

The material of Book I has been carefully selected and so graded that the student can pass from the Introduction to real constructive work of his own as smoothly as possible. Here and throughout the other four books the proofs of nearly all the theorems are complete. In every demonstrated theorem a clear statement of the "Method" of the proof precedes the proof itself. The plan of the proof is thus apparent in advance. The psychological value of this innovation is obvious.

Work with the straightedge, compasses, and protractor, which has been begun in the Introduction, is continued in Book I. The various constructions needed in drawing the

figures for the theorems are given just before they are first required. In this manner work in constructions is systematically carried on with the work of the theorems.

In each of the five books the traditional list of theorems is given. Numerous simplifications in the proofs themselves have been made, as well as some desirable changes of order. Extensive lists of review questions are included at appropriate points. At the end of each book is given a summary of the work covered.

Important and suggestive work on the locus is given toward the end of Book II. Graphs of the equations of the straight line and the circle are given as examples of loci and illustrate a very important relation between algebra and geometry.

A simple introduction to numeric trigonometry is included in Book III. This gives an interesting application of similar triangles and throws light on a fundamental branch of mathematics.

In Books IV and V, especially, considerable work on mensuration is given, thus emphasizing important practical applications of geometry. A list of practical problems appropriate for each of the five books will be found in a supplement at the end of Book V. Notes at suitable points give the student interesting glimpses of the history of geometry.

The material included in the text conforms to the recent important recommendations of the various examining bodies.

Suggestions for a Minimum Course

A minimum course has been outlined for those teachers who may find some omissions necessary and who desire suggestions as to what work is fundamental. For such teachers the following list of pages which may be omitted has been prepared. This short course is based on the requirements of the College Entrance Examination Board.

Introduction. Omit no pages.

Book I. Omit exercises on pages 94 and 106–107, pages 120–122 (except § 151), and exercises on pages 138–141.

Book II. Omit pages 170, 183–186 (except §§ 224–225), 200–203, 209, 212–214, 221–222.

Book III. Omit pages 237, 268, 269, 278–282 (except § 304), 285, 286, 289–292, 296–297, 298–308.

Book IV. Omit pages 323 (except § 332), 332, 333, 339, 348–350.

Book V. Omit pages 381, 386–396, 398, 402–405.

Applications. Omit pages 407–417.

To make the foregoing course effective numerous exercises on each of the theorems and constructions must be made an essential part of the work. If at all possible, it is desirable that for the more able students this minimum course be supplemented with some of the omitted work.

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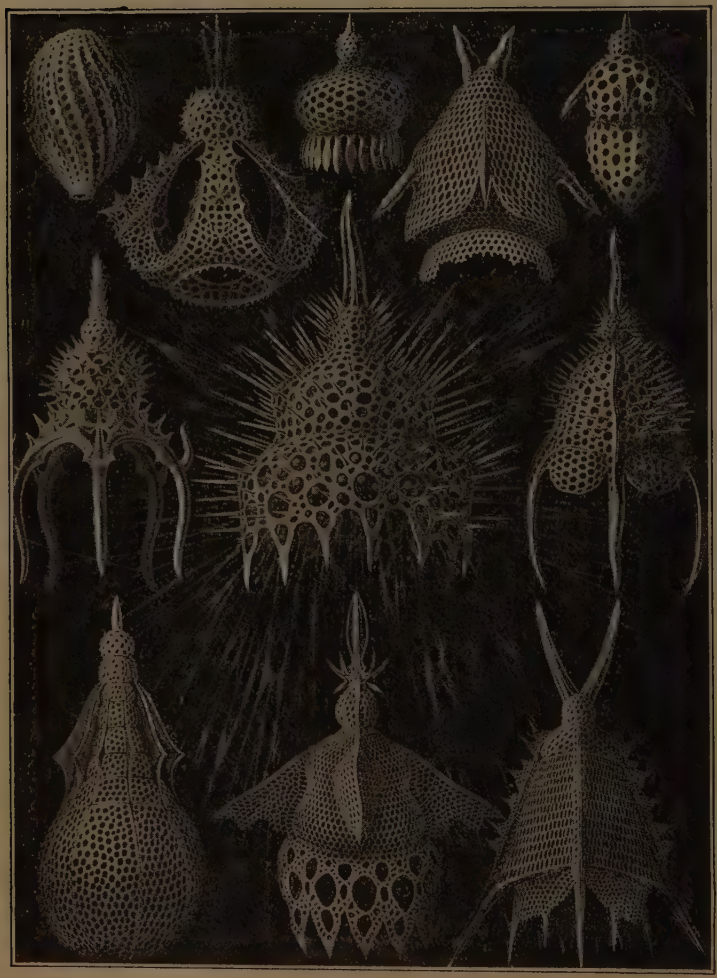


The skeletons of small animals living in sea water furnish beautiful examples of the way geometric patterns appear in nature

Courtesy of the American Corporation

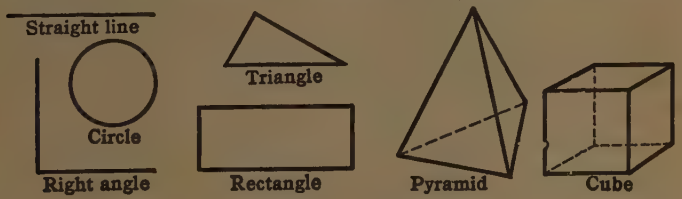
NEW PLANE GEOMETRY

INTRODUCTION

FIRST VIEW OF GEOMETRY

It is the glory of geometry that from so few principles, fetched from without, it is able to accomplish so much. — NEWTON

1. Discussion. The student has learned in arithmetic and in algebra something of the nature and use of geometry. Certain facts about the straight line, the right triangle, the rectangle, the circle, the pyramid, and the sphere have been met before. That a line has one dimension, length; a surface two dimensions, length and breadth;



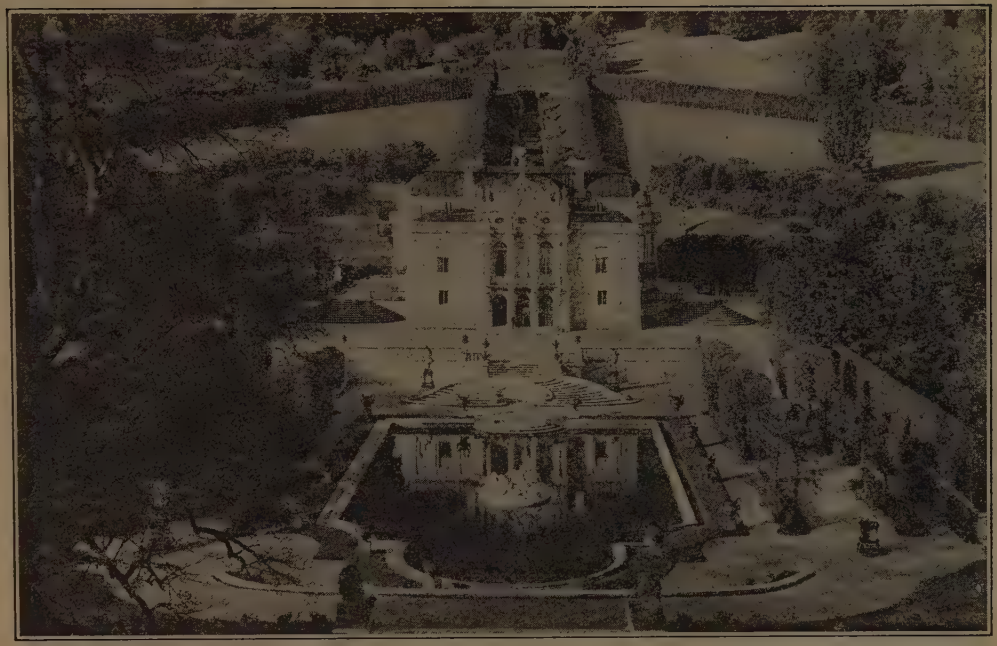
and a solid three dimensions, length, breadth, and thickness, are important ideas which are already familiar to us. Geometry uses the numbers and methods of arithmetic and the symbols and equations of algebra, but its main concern is the study of some of the useful and interesting characteristics of geometric figures (angles, triangles, circles, cubes, etc.), — figures made up of points and lines or bounded by lines and surfaces. Up to the present time

the student has taken for granted the truth of such statements as (1) the area of a triangle is one half the product of the base by the altitude; (2) the length of a circle is $2\pi r$; (3) the square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides. In his study of plane geometry the student will learn to prove these and many other statements of this kind.

2. How geometric truths are proved. There are many reasons why we believe certain things to be true. We accept the word of scientists that the blood circulates through the body, returning again and again to the heart, or that an artificial supply of oxygen is needed in climbing to very great heights, as in the ascent of Mt. Everest and in altitude flights in airplanes. Statements like these we believe to be true because someone we think ought to know tells us they are true; we have not the means of proving them for ourselves. Still other things, however, we can prove for ourselves by starting with simple facts to which we all agree. Then by easy logical steps, each of which can be fully understood, we arrive at results that we did not know before. We shall find that the truths of geometry, such as those referred to in § 1, are of this sort; we shall learn to work them out for ourselves, and convince ourselves of their truth.

3. Uses of geometry. A great many geometric figures are both useful and interesting. To find out more about these figures and to learn some of the uses to which they can be put are among the important purposes of geometry.

Geometry has many other uses, a few of which may be mentioned here. The student can, of course, think of more. Still others will be discovered from time to time as we go ahead with the study of plane geometry. A large number of the advances of our modern civilization have



This picture gives an example of the use of geometry in the building itself and in the various features of the setting in which it has been placed

Snowflakes furnish beautiful and striking geometric patterns. Ice crystals tend to form hexagonal shapes, as shown here

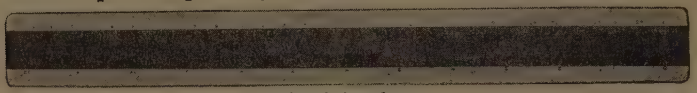
come about through the application of sciences which are based on geometric knowledge; one needs to know geometry to understand the principles of aëronautics, television, or any of our newer inventions. We must know geometry to become engineers, bridge-builders, or architects. We cannot study forestry, city-planning, aviation, surveying, or machine work and design without first learning geometry. Many of the charts and diagrams that are used by business men, by nurses, by navigators, and by countless other men and women can be better understood and more efficiently used if we know geometry.

Geometry helps us, therefore, to know and to understand the interesting scientific discoveries that are being made, and is useful in many of the occupations, trades, and professions by which men and women earn their livelihood. You will find as you go on with your studies in science that geometry, together with algebra, forms the foundation for the physical sciences and is of great importance in statistical subjects.

4. Other reasons for the study of geometry. Geometric figures are very common in artistic designs. In clothing, rugs, curtains, art windows and window decorations, in the planning of buildings, and in the laying out of parks and playgrounds, the use of geometric figures has long been extensive. In nature, too, there are remarkable exhibitions of geometric patterns in the symmetric figures of leaves and trees, in the shapes of snowflakes and crystals, in the cells in which bees store their honey, in the eye of the fly, and in countless microscopic forms of life. The earliest American mathematician of distinction was accustomed to remark, "God is the great geometer." The student will find a greater pleasure in the beauty of form in nature and in art if he knows geometry.

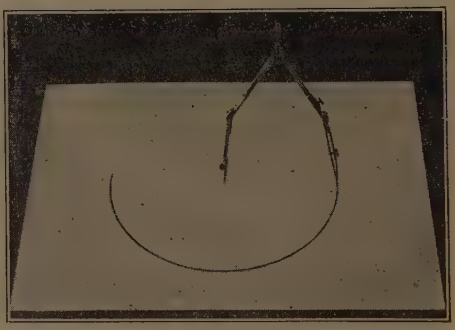
Lines and Instruments of Geometry

5. Divisions of geometry. Geometry is divided naturally into two parts: *plane geometry*, which treats of certain figures that may be drawn on a flat (plane) surface, and *solid geometry*, which treats of figures having three dimensions.



Straightedge

6. Lines used in geometry. Plane geometry deals with *straight lines* and *circles* and with the figures formed by straight lines alone or by straight lines and circles or parts of a circle. In advanced courses in geometry many other curves are studied.

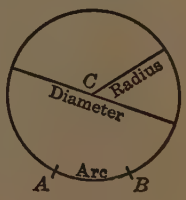


Compasses

7. Instruments used in geometry. Since we are limited in elementary geometry to straight lines and circles, we make use only of the instruments required to draw them; namely, the *straightedge* and the *compasses*.

The straightedge is used for drawing straight lines.

The compasses are used for drawing circles and to make a line in one part of a figure equal a line in another part.



8. Circle. A *circle* is a closed plane curve every point of which is equally distant from a point in the plane of the curve called the center.

9. Radius. A *radius* of a circle is a straight line from the center to the circle.

10. Diameter. A *diameter* is a straight line through the center and having both ends in the circle.

11. Arc. An *arc* is a portion of a circle, as AB.

EXERCISES IN GEOMETRIC DRAWING

1. Draw a circle whose radius is 1 in.

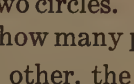
2. Draw a line AB 2 in. long. Then, with A and B as centers and with a radius of $1\frac{1}{2}$ in., draw two circles. Do the two circles cut (intersect) each other? In how many points?

3. Draw two circles intersecting each other, the radius of one being 1 in. and the radius of the other being 2 in.

4. Draw any two circles which intersect. How many points of intersection are there?

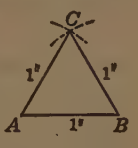
5. May two circles have only one point in common? If so, draw two such circles.

6. Given the lines a and b, construct by means of compasses and straightedge a line equal to $a + b$ and another equal to $b - a$.



7. Draw a circle and a straight line intersecting it. How many points of intersection are there?

8. Draw a line AB 1 in. long. With A as a center and with a radius of 1 in. draw an arc. With B as a center and with the same radius draw an arc intersecting the first at C. Draw AC and BC. Are the three sides of this triangle equal?



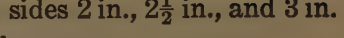
9. Draw a triangle each of whose sides is $1\frac{1}{2}$ in.

10. Draw a triangle whose sides are 1 in., $1\frac{3}{4}$ in., and $2\frac{1}{4}$ in.

11. Draw a triangle as before, making AB equal to 1 in. and AC and BC each equal to $1\frac{1}{2}$ in.

12. Construct a triangle with sides 2 in., $2\frac{1}{2}$ in., and 3 in.

13. Given the lines a, b, and c, construct a triangle having as sides lines equal to a, b, and c.



14. Given a, b, and c of Exercise 13, by means of the straightedge and compasses construct lines equal to $2a$, $2b$, and $2c$, and construct a triangle having these lines as sides.

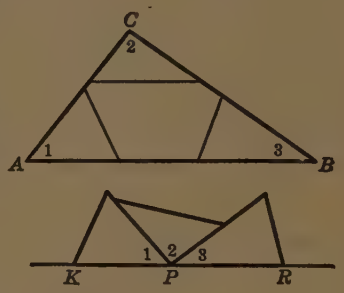
15. Draw a triangle having as sides $3a$, $2b$, and c.

16. Draw a triangle having sides equal to $2a$, $2b$, and c.

17. Can you draw a triangle whose sides are $2\frac{1}{2}$ in., 3 in., and 6 in.? If not, why not?

18. Construct a triangle with sides 3 in., 4 in., and 5 in.

12. Experimental geometry. It is possible to arrive at some geometric facts experimentally. For example, the sum of the angles of any triangle may be found in the following manner: Make a cardboard triangle ABC and cut off the three corners along the light lines as indicated in the adjacent figure. Then draw a straight line KR and place the three pieces as indicated. If this is carefully done the sum of the three angles of the triangle, now that the vertices A, B, and C are at the point P, exactly fills up the space around P on one side of KR.



From one point of view the straight line KPR does not seem to resemble an angle; but since the sum of the three angles of the triangle just fits on one side of the line, we call KPR a *straight angle*. We may say, therefore, that the sum of the angles of the triangle ABC is a straight angle.

EXERCISES

1. Cut out a cardboard triangle whose sides are about 4 in., 5 in., and 6 in. Then cut off the corners as described on page 6, and show that the sum of the three angles of the triangle is a straight angle.

2. Perform Exercise 1 with a triangle whose sides are 4 in., 4 in., and 5 in.

3. Construct and test in this way three triangles of different shapes.

If we used many triangles of different shapes, the result would always be the same. We could thus, with considerable labor and some certainty, arrive at the fact that the sum of the angles of any triangle is equal to a straight angle, or at any rate very nearly equal to one. If we looked casually at triangles of different shapes, we should probably not guess that the sum of the three angles is exactly the same in each case; and even if we did surmise that the sum of the angles is the same for all triangles, we should hardly imagine that it would turn out to be exactly a straight angle in every case. Yet this is the fact, and it is unexpected results like this that the study of geometry enables us to reach.

13. The protractor. The experimental method in geometry may be carried farther by using an instrument for measuring angles, called a *protractor*. It can be used for two purposes:

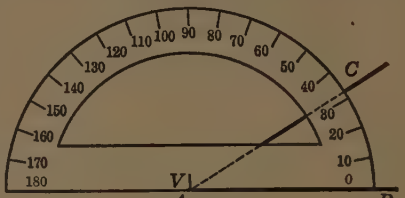
(*a*) to measure the size of (that is, the number of degrees in) a given angle;

(*b*) to construct an angle of a given size (that is, containing a certain number of degrees).

While it is true that only the straightedge and compasses are actually required in geometry, other instruments, of which the protractor

and the ruler are two, are an aid to a clearer understanding of the subject; for, just as the ruler with inch marks on it enables us to tell the length of a line (that is, to measure it), so the protractor enables us to measure angles and to construct angles of any desired size.

14. Measuring an angle with a protractor. To measure an angle BAC place the point V of the protractor on the vertex A of the angle, and the zero point on line AB. The point where the side AC crosses the scale determines the number of degrees (32) in the angle.



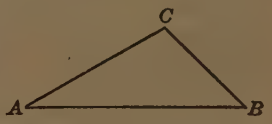
Protractor

We may use the protractor to show that the sum of the angles of a triangle is 180°, by cutting out cardboard triangles of various sizes and measuring each angle separately and then finding the sum of the three angles, thus verifying the result of the exercises on page 7.

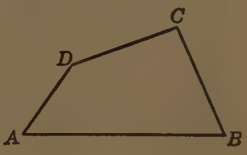
EXERCISES WITH A PROTRACTOR

1. Draw an angle about like ABC and measure it with the protractor.

2. Draw a triangle shaped like ABC and measure each angle. Find the sum of the three angles.

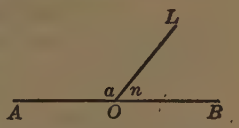


3. Draw a four-sided figure like $ABCD$ and measure each of the four angles. Find the sum of the four angles.



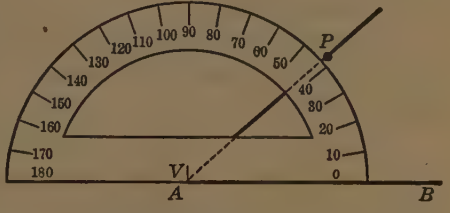
4. In a figure like the adjacent one measure angle a ($\angle a$) and angle n ($\angle n$). What is their sum?

5. Suppose AOB in the adjacent figure is a straight line and $\angle a = \angle n$. How many degrees would there be in each?



6. If $\angle a = \angle n$ in the figure of Exercise 4, each is called a *right angle*. How many degrees in a right angle? What is the relation of a straight angle to a right angle?

15. Constructing with a protractor an angle of a given size. An angle of 42° can be constructed as follows: Draw a line AB and place the point V of the protractor at A and the zero point of the protractor on AB. Mark the point P opposite the 42° mark of the scale. Then remove the protractor and draw the line AP. The angle BAP will contain 42°.



EXERCISES WITH A PROTRACTOR

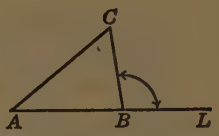
1. Use the protractor and draw an angle of 54°; of 81°; of 143°; of 210°.

2. Draw two lines, AB and LM, intersecting at O. Then measure angles AOL and BOM. Compare results. Measure also angles AOM and BOL and compare results.

3. Perform Exercise 2 again, drawing lines to intersect at a different angle.

4. The two pairs of angles measured in Exercises 2 and 3 are called *vertical* angles. What seems to be true as to the size of vertical angles?

5. Draw a figure like the adjacent one and measure the three angles A, C, and LBC. Compare the sum of angles A and C with angle LBC.

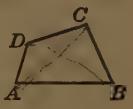


6. Perform the work of Exercise 5 with a new figure, having triangle ABC of a different shape.

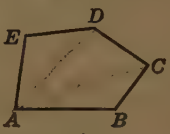
7. The angle LBC is called an *exterior* angle of the triangle ABC, and angles A and C are called the opposite *interior* angles. What relation seems to hold between an exterior angle of a triangle and the sum of the opposite interior angles?

8. Draw a large triangle, measure each of its angles with the protractor, and determine whether their sum is a straight angle, or 180°.

9. Draw any four-sided figure, measure each angle, and determine whether their sum is two straight angles, or 360°.

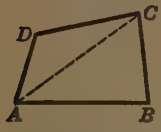


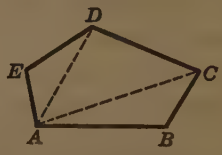
10. Draw any five-sided figure, measure each angle, and find their sum. How many degrees is the sum? How many straight angles?

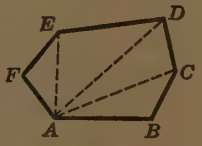


11. Perform the work of Exercise 10 for a six-sided figure.

In order to discover a more general truth consider the following polygons (flat figures bounded by straight lines):







12. Is the sum of the angles of triangles ABC and ACD equal to the sum of the four angles of the polygon $ABCD$?

13. Is the sum of the angles of the triangles in the polygon *ABCDE* equal to the sum of its five angles?

14. Answer the question of Exercise 13 for the polygon *ABCDEF*.

15. Try to fill out correctly the blanks in the following:

The sum of the interior angles of

a polygon of 3 sides is ____ straight angles;
a polygon of 4 sides is ____ straight angles;
a polygon of 5 sides is ____ straight angles;
a polygon of 6 sides is ____ straight angles;
a polygon of 7 sides is ____ straight angles;
a polygon of 8 sides is ____ straight angles;
a polygon of n sides is ____ straight angles.

The preceding exercises with the protractor indicate, but do not prove, the truth of the following:

(*a*) Vertical angles are equal.

(*b*) An exterior angle of a triangle equals the sum of the two opposite interior angles.

(*c*) The sum of the angles of any triangle is one straight angle.

(*d*) The sum of the interior angles of any polygon having n sides is $(n - 2)$ straight angles.

16. Construct a triangle having each side 3 in. Measure each angle with the protractor. Are the three angles equal?

17. Construct a triangle having sides $3\frac{1}{2}$ in., and measure each angle as before. Compare results with those just obtained.

18. Construct a triangle whose sides are 3 in., 3 in., and 4 in. Measure the three angles with a protractor. Are any of the angles equal? How many? Is the angle opposite side 4 equal to any other angle of the triangle?

19. Construct a triangle with sides 4 in., 4 in., and 3 in., and measure each angle. Are any of the angles equal?

How many? How many sides of this triangle are equal? How are the equal angles situated with respect to the equal sides?

20. Construct a triangle with sides 3 in., $3\frac{1}{2}$ in., and 4 in., and measure the three angles. Are any of the three angles equal? If so, which ones?

21. Construct another triangle with sides $2\frac{1}{2}$ in., 3 in., and $3\frac{1}{2}$ in. Measure the angles. Are any of the angles equal? If so, which ones?

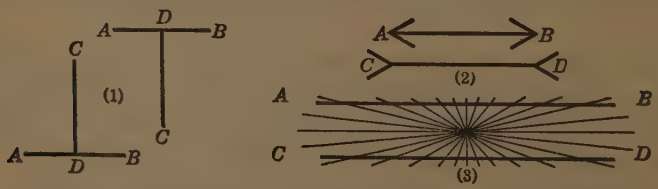
22. Make the following statements agree with the preceding results by putting in the missing word:

(*a*) If three sides of a triangle are equal, its angles are ____.

(*b*) If two sides of a triangle are equal, the angles opposite them are ____.

(*c*) If the three sides of a triangle are unequal, the three angles of the triangle are ____.

16. Defects of the experimental method. One may easily make a mistake by concluding that a geometric figure has certain lines in it equal, or has certain angles equal, or has certain relations between its lines or its angles because to the eye such relations appear to be present. The following figures show how easily the eye may be deceived:



QUERY 1. In figure (1), is AB equal to CD? In figure (2)?
QUERY 2. In figure (3), are AB and CD straight lines?

The experimental method also fails in accuracy. For example, angles of a triangle cannot be measured exactly enough to prove that the sum of its angles is precisely 180°, and neither more nor less than that.

The accuracy with which the student can carry out these measurements would not satisfy a watchmaker or the designer of a fine machine. It often suggests a fact and verifies it up to a certain point, but with imperfect instruments perfection cannot be assured. Aside from this, the experimental method also fails in certainty. Even if the accuracy were perfect, the fact that the sum of the angles of each of ten or even a hundred different triangles is found to be 180° does not prove that the sum of the angles of every possible triangle is two right angles, and neither more nor less than 180°. In other words, many numeric verifications of a statement at most merely render its truth a little more probable but do not really prove that it is true in all possible cases. The following exercises, based on a simple algebraic formula and some arithmetic, illustrate a failure of the experimental method.

The number 7 is a prime number, because no integer except itself and the number 1 will exactly divide it.

Similarly, the numbers 2, 3, 5, and 11 are prime.

EXERCISES

1. What are the prime numbers between 10 and 40?

2. What are the prime factors of 30? of 48? of 60? of 72?

3. Make a list of all the prime numbers less than 100.

4. Are the numbers 113, 131, 151, 173, 197, 223, and 251 prime numbers?

About 300 B.C. Euclid proved that the total number of prime numbers is countless. It has long been known that prime numbers occur irregularly. There are 169 prime numbers which are less than 1000. Between the numbers 1000 and 2000 there are 135 primes; between 2000 and 3000 there are 127; between 8000 and 9000 there are 110; between 9000 and 10,000 there are 112. For many centuries mathematicians have studied prime numbers, hoping to find out the law underlying their occurrence (that is, they have sought to discover a simple method or formula for determining the next prime number after a given one), but so far without success. Many approximate formulas for primes have, however, been devised. One of these is $n^2 + n + 41$.

5. Is $n^2 + n + 41$ prime when $n = 1$? when $n = 2$? when $n = 3$? when $n = 4$?

6. Fill out the blanks in the following table:

$n =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$n^2 + n + 41 =$	43	47	53	61	?	?	?	?	?	?	?	?	?	?

7. Compare the lower line of the completed table with the results of Exercises 3 and 4. Is the value of $n^2 + n + 41$ a prime number for all values of n from 1 to 14?

From the completed table and the results of Exercises 3 and 4 it is clear that if n is any integer from 1 to 14 inclusive, the value of $n^2 + n + 41$ is a prime number.

It can be shown, moreover, that the formula holds for many numbers beyond 14. Actually it gives prime numbers for every value of n from 1 to 39 inclusive. It fails for 40, however, since $(40)^2 + 40 + 41 = 1681 = (41)^2$. Since $(41)^2 + 41 + 41 = (41)(43) = 1763$, the formula fails for 41 also.

Here, then, is a formula for primes which, when tried, holds thirty-nine times but fails the fortieth time. This shows us that many numeric verifications of any statement do not prove it true in all cases.

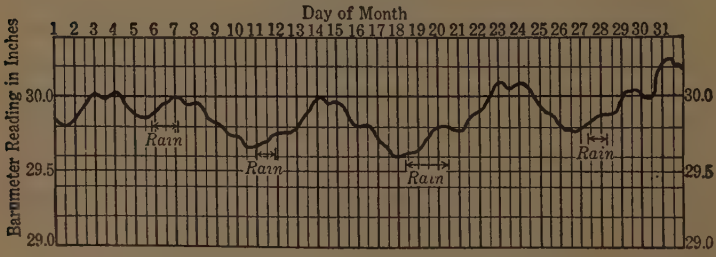
In many branches of science the experimental method, though often laborious and slow, is the method by which the most progress is made. Sometimes it is the only method possible. Even in mathematics it has been of some value. But the facts of geometry are so many and varied that a method is desirable which will quickly prove that a geometric statement is exactly true, and always true. The method which does this is, as we shall see, the *demonstrative* method, which applies to algebraic as well as to geometric facts.

17. Two types of reasoning. Sometimes a number of facts can be grouped together and from them we can make a guess or inference as to a new fact. An example of this type of reasoning arose a while ago in the case of the cotton crop. Investigators studied the records of rainfall, of temperature, of barometric pressure, and of the amount of the cotton crop over a period of several years. They made the inference that heavy winter rains, followed by warm, sunny spring weather produced the best cotton crops. To find out whether their inference was correct, they observed all the conditions of climate and weather and tried to see if they could predict what the cotton crop would be. As they were able to predict quite accurately, they felt satisfied that their inference was correct. This is the so-called "scientific method." When we use this method we study groups of facts and from them we make a guess or inference. Then we experiment or observe to see whether our inference enables us to predict what will happen under similar conditions. Even then we cannot be absolutely certain of the correctness of our inference, for some important facts may have been overlooked.

It is often possible to use a sort of reasoning in which this danger does not arise. This is done in suitable cases

by beginning with an assumption which we feel confident is true. We then go on to the facts which follow from the assumption, finding one or more new facts as the result of the process. For instance, if a detective has traced a man into a certain house, and has had the doors and windows continuously and securely guarded since the man's arrival, he is sure the man is still in the house, because he assumes that one can leave that house only by a door or a window. If his assumption is correct, he can be absolutely sure that the result of his reasoning is correct also. If his assumption is wrong, and there is a secret underground exit from the house, his conclusion *may* also be wrong. (Of course the man may not have used the secret exit.)

EXERCISES



Barometer readings, at noon, for one month

1. How many times during the month for which the records are given did it rain? On what dates? On what dates were the barometric readings high? On what dates were they low? What relation do you see between these sets of facts?

NOTE. We must be careful to put our statement of this relation in the form "It seems that . . ." or some similar form. We cannot be sure that our inference is right for all cases.

2. Mary's dress costs more than Helen's; Alice's dress costs more than Mary's. What relation can you state between the cost of Helen's dress and Alice's dress?

3. Poor eyesight is a great handicap in school work. Properly fitted lenses will help to relieve poor eyesight. What relation can you state between school work and properly fitted lenses for students with poor eyesight?

4. The following table shows the length of time required for a weight swinging on a cord to make one complete vibration:

Weight	Length of Cord	Time of Vibration
10 grams	50 cm.	1.4 sec.
20 grams	200 cm.	2.8 sec.
20 grams	50 cm.	1.4 sec.
10 grams	100 cm.	2.0 sec.
30 grams	100 cm.	2.0 sec.
10 grams	200 cm.	2.8 sec.
30 grams	200 cm.	2.8 sec.
30 grams	50 cm.	1.4 sec.
20 grams	100 cm.	2.0 sec.

Arrange in groups all the lines of the table that show the same time of vibration. Does the time of vibration seem to have anything to do with the weight that is swinging back and forth? Does the time of vibration seem to have anything to do with the length of the cord from which the weight swings? State your inference from these groups of facts.

5. Which of the above exercises are examples of scientific reasoning? Which are examples of reasoning from assumptions?

18. Reasoning in geometry. The discovery of a way of arriving at sure results by reasoning from known truths is one of the most important developments of the human mind. In geometry more than in any other subject of

study this process is explained and illustrated. Important as are the practical applications of geometry in the field of engineering, of commerce, and of navigation, the application of this kind of reasoning in the world of each individual's thinking is even more important; for not everyone is interested in building bridges and designing machines, but every person is interested in being able to recognize and to use clear straight thinking.

19. Assumptions in geometry. Whenever reasoning based on assumptions is used, whether in everyday life or in mathematics, one must start with assumptions which everyone believes to be true. If the assumptions are not true, of course the results will be in doubt; if the assumptions are entirely true, one may be absolutely confident of the results.

One of the most important preliminary steps in the proof of geometric truths is the clear statement of the assumptions with which we begin our study. These assumptions will naturally have to do with the nature of lines, circles, and angles, and the numbers by means of which they are measured. As an illustration of this method we now present an assumption about the straight line, and show how from this assumption a number of different results are derived.

20. Fundamental assumption. Between two points one and only one straight line can be drawn.

EXERCISES

1. Given a number of points no three of which are in a straight line, draw all the lines possible each of which passes through two of the points. How many lines are there (*a*) when the number of points is 2? (*b*) when the number of points is 3? (*c*) when the number of points is 4?

2. Show that for 5 points the number of lines is 10.

3. Show that the number of lines is 15 if the number of points is 6.

A general law underlies such results as the preceding. Instead of drawing a figure and counting the lines, Exercise 3 might be solved thus: From every one of the 6 points a line can be drawn to each of the other 5. This can be done six times, — once for each point. This appears to make a total of 30 lines. But every line thus drawn is counted twice, for a line drawn from point A to point B is later drawn a second time from point B to point A. Hence the total number of distinct lines is half the total number, which is $\frac{6 \times 5}{2}$, or 15. Similarly, for 7 points the number of lines is $\frac{7 \times 6}{2}$, or 21, and for 8 points the number of lines is $\frac{8 \times 7}{2}$, or 28. The results may be summarized in the following table:

Number of points . .	2	3	4	5	6	7	8	9	10	n
Number of lines . . .	1	3	6	10	15	21	28	?	?	?

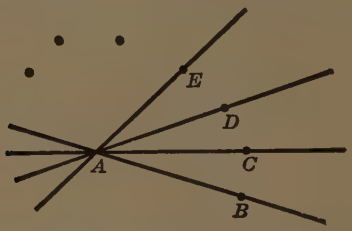
EXERCISE

Fill out the blanks in the table by inspection without looking at the next section. Is the expression for the number of lines corresponding to n points a general formula?

21. Proof of a geometric proposition. The theorem to which the work of the preceding section leads will now be stated and its formal proof given.

Theorem

The number of straight lines determined by n points no three of which are in a straight line is $\frac{n^2 - n}{2}$.



Given n points, $A, B, C, D, E, \cdots$, no three of which are in a straight line.

To prove that the number of possible lines each passing through a pair of points is $\frac{n^2 - n}{2}$.

Proof

Statements	Reasons
1. From point A a line can be drawn through every other point.	1. Between two points one straight line may be drawn.
2. The total number of lines thus drawn through A is $n - 1$.	2. There are $n - 1$ points, excluding point A.
3. Similarly, from each of the other points, B, C, D, etc., $n - 1$ lines can be drawn.	3. Same reasons as for steps 1 and 2.
4. Hence the total number of lines so drawn is $n(n - 1)$.	4. The sum of n numbers, each equal to $n - 1$, is $n(n - 1)$.
5. But every line is thus drawn twice.	5. Once, for example, from A to B, and again from B to A.

6. Therefore the total number of lines is $\frac{n(n-1)}{2}$, or $\frac{n^2-n}{2}$.	6. From steps 4 and 5.
7. These lines are all distinct.	7. No three of the n given points are in a straight line.

NOTE. If in this formula, $\frac{n^2-n}{2}$, we now substitute for n the numbers 2, 3, 4, etc., we obtain the results previously found in § 20.

ASSUMPTIONS, DEFINITIONS, AND PRELIMINARY THEOREMS

22. The basis of reasoning. In any intelligent discussion two things are necessary. In the first place, everybody concerned must use words in the same sense; for, evidently, if two persons mean different things by the same word, or if one of them has no idea of the meaning of certain words that are used, it is impossible for them to agree. If a conclusion appears to be reached in such a case, it is of no value.

Applied to the study of geometry this means that the definitions of the words used must be clearly understood if we expect to understand the subject. The idea conveyed by a definition must be sharp and clear, so that we shall recognize the thing defined and be able to give in our own words a concise and correct definition.

The second thing necessary for a clear discussion is that we should know what may be assumed at the start. If two people are to reach an agreement in a conference, they must find common ground, and both of them must know just what that common ground is. In all probability most of the misunderstandings and disagreements in the world occur just because people do not take pains to decide in the beginning what they can agree upon.

So in geometry we need to state clearly the assumptions that we take as a starting point for our study of the subject. Every conclusion that we may reach comes back to these agreements and to the definitions of the terms used. Thus the study of geometry may suggest how to handle problems which arise in everyday life.

23. Assumptions, axioms, and postulates. One important problem concerns the assumptions which are needed to begin this introduction to geometry. It is customary to call the assumptions that have to do with the *arithmetic aspect* of geometry, *axioms*. The assumptions that are *purely geometric* are often called *postulates*.

An example of an axiom is: *Equal numbers added to equal numbers give equal numbers.*

An example of a postulate is: *Through two points one straight line and only one can be drawn.*

The first of these applies to algebra and arithmetic (that is, to numbers) while the second deals with a geometric notion. Many of the axioms needed have already been met and used in algebra. For convenience of reference the more important axioms and postulates are given on pages 35–36.

It is necessary to understand clearly the meaning of the assumptions. This is not difficult, however, as is obvious from the fact that they used to be called "self-evident truths."

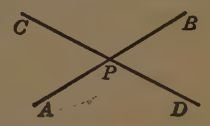
24. Definitions. The next problem is to choose the geometric terms needed and to give their meaning. Where definition is difficult, as with the terms *point* and *line*, the ideas which they denote can be brought out by illustration. A certain order is often essential in stating definitions. For example, the word *degree* cannot be easily defined before the terms *angle* and *right angle* are understood.

25. Fundamental geometric concepts. The simplest notions in geometry are points, straight lines, and angles.

26. Point. A *point* is represented to the eye by a dot. It has position but not length, breadth, or thickness. The geometric points of which we are thinking in the figures that we draw will be denoted sufficiently well by the dot made by a sharp pencil.

27. Straight line. If a weight is suspended by a fine thread, the thread is then a good illustration of a *straight line.*

If two lines cross or intersect, the part common to the two is a point.



Thus, P is the point of intersection of the straight lines AB and CD.

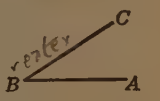
A line may be thought of as having length but no breadth or thickness.

The geometric straight line extends indefinitely far in both of its directions. It has no ends at all. Any portion of a straight line may be represented by, or drawn with the help of, a straightedge.

A line of definite length is called a *line segment.*

When the word *line* is used alone, a straight line, not a curved line, is meant.

28. Angle. An angle (symbol $\angle$) is the figure formed by two line segments which meet each other at a point.



In the adjacent figure, ABC is the angle, AB and CB are the *sides* of the angle, and B is the *vertex.*

29. Plane. A *plane* or a flat surface is a surface such that a straight line joining any two of its points lies wholly in or on the surface.

In finishing up paving or sidewalks a straightedge is often laid across it in several directions to test its flatness.

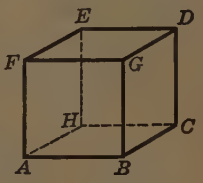
Surfaces have two dimensions: length and breadth.

30. Geometric figures. A point, a line, a surface, or any combination of these is called a *geometric figure.*

A plane figure is any geometric figure which lies in a plane (flat) surface.

Most of the figures of this text are supposed to be in a plane surface, which we call the plane of the paper.

We may, however, represent by a drawing in a plane a block like $ABCD$, which is a geometric figure called a *solid.* A solid has three dimensions: length, breadth, and thickness.



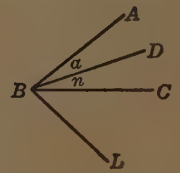
The smooth faces $ABGF$, $BCDG$, $GDEF$ are surfaces; the edges AB, BC, CD, etc. are line segments; and the corners where three edges meet, like A, B, C, are points.

The boundaries of a solid are surfaces, the boundaries of a surface are lines, and the boundaries of a line segment are points.

31. Straight angle. A *straight angle* is an angle whose sides extend in opposite directions from the vertex and form a straight line.

32. Straight-angle postulate. *All straight angles are equal.*

33. Adjacent angles. Two angles which have the same vertex and a common side between them are called *adjacent* angles.



Thus, $\angle ABC$ and $\angle CBL$ are adjacent angles.

Adjacent angles may be added or subtracted as follows:

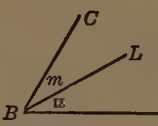
$$\angle ABD + \angle DBC = \angle ABC,$$

and

$$\angle ABL - \angle ABD = \angle DBL.$$

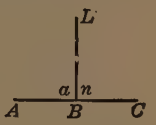
When several angles are adjacent, they may be more conveniently referred to by a single letter or a number placed within the angle near the vertex than by three letters. Thus, $\angle ABD$ and $\angle DBC$ may be called $\angle a$ and $\angle n$ respectively. When three letters are used to name an angle, the vertex letter must come between the other two.

34. Postulate. *An angle can be bisected by one and only one line.*



Thus, if BL bisects $\angle ABC$, $\angle m = \angle a$, since they are halves of the whole angle ABC.

35. Perpendicular. If one straight line meets another so as to make the two adjacent angles equal, the two lines are *perpendicular* (symbol $\perp$) to each other.



The symbol $\perp$ means "perpendicular" or "is perpendicular to."

Thus, if $\angle a = \angle n$, the line LB is perpendicular to AC.

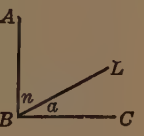
Since an angle has only one bisector, there is only one line that bisects the straight angle ABC and makes $\angle a$ equal to $\angle n$. This makes clear the following postulate: *At a given point of a line there is one perpendicular to it and only one.*

36. Right angle. A *right angle* is an angle whose sides are perpendicular each to each. Any two right angles are equal to each other, since a right angle is half a straight angle.

In the figure in § 35, $\angle a$ and $\angle n$ are right angles, and $\angle a + \angle n =$ a straight angle; that is, two adjacent right angles equal a straight angle.

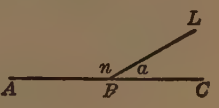
37. Degree. If a right angle is divided into ninety equal angles, each part is an angle of one *degree* (1°).

38. Complementary angles. One angle is the *complement* of another if their sum is a right angle.



Thus, if $\angle ABC$ is a right angle, $\angle a$ and $\angle n$ are complementary.

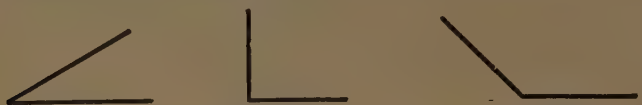
39. Supplementary angles. One angle is the *supplement* of another if their sum is a straight angle (or 180°, by §§ 35, 36, 37).



Thus, if $\angle ABC$ is a straight angle, $\angle n$ and $\angle a$ are supplementary.

40. Acute angle. An angle less than a right angle is an *acute angle.*

41. Obtuse angle. An angle greater than a right angle and less than a straight angle is an *obtuse angle.*



EXERCISES ON COMPLEMENTARY AND SUPPLEMENTARY ANGLES

1. What is the complement of 72°? 64°? 31°? 12°? $x°$? $(a - x)°$?

2. What is the supplement of 16°? 40°? 60°? 75°? 110°? 150°? 90°? 0°? $x°$? $(90 - x)°$?

3. What kind of angle do the hands of a clock make at 3 o'clock? 1 o'clock? 2 o'clock? 4 o'clock? How many degrees in each?

4. Through how many degrees does the minute hand move in 1 hr.? the hour hand?

5. How many degrees are there in the angle made by the hands of a clock at half past 12? 20 min. past 12?

6. Through how many degrees does the minute hand move in 5 min.? 10 min.? 20 min.? 30 min.? 2 hr.? 3 hr. and 5 min.?

7. Through how many degrees does the hour hand move in 5 min.? 10 min.? 15 min.? 35 min.? 1 hr.? 2 hr.? 1 day, or 24 hr.?

8. The complement of an angle is twice the angle. Find the angle.

HINT. Let x = the number of degrees in the angle. Then $90 - x =$ the number of degrees in the complement; then $90 - x = 2x$; etc.

9. An angle is four times its complement. Find the angle.

10. The supplement of an angle is four times the angle. Find the angle.

HINT. $180 - x = 4x$.

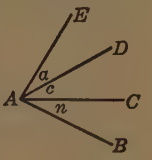
11. The complement of an angle is one fourth of its supplement. Find the angle.

12. Two angles are supplementary, and one is 37° greater than the other. Find both angles.

13. The supplement of an angle is three times its complement. Find the angle.

14. The supplement of an angle is five times its complement. Find the angle.

15. The complement of an angle is one sixth of its supplement. Find the angle.



16. Name by three letters the angle equal to

$\angle BAD + \angle DAE$, $\angle BAD - \angle CAD$,

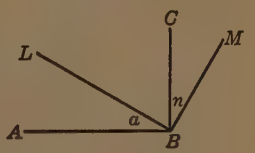
$\angle CAD + \angle DAE$, $\angle BAE - \angle DAE$.

17. Name by three letters the angle equal to

$\angle a + \angle c$, $\angle c + \angle n$, $\angle a + \angle c + \angle n$.

18. If, in the accompanying figure, $\angle a = \angle n$, show that $\angle x = \angle z$.

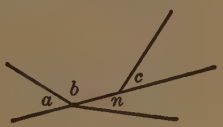
19. In the adjacent figure, $\angle ABC$ and $\angle LBM$ are right angles. Show that $\angle a = \angle n$.



20. In the accompanying figure state which of the following pairs of angles are adjacent:

$\angle n$ and $\angle c$, $\angle a$ and $\angle c$,

$\angle a$ and $\angle n$, $\angle a$ and $\angle b$.



42. Proposition. A *proposition* is the general name for any assertion.

Thus, "Still water runs deep" is a proposition.

43. Theorem. A *theorem* is a mathematical proposition to be proved.

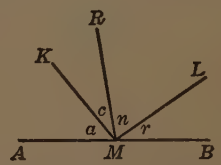
An example of an algebraic theorem is: *The square of the sum of two numbers equals the sum of their squares plus twice their product.*

An example of a geometric theorem is: *If two sides of a triangle are equal, the angles opposite them are equal.*

44. Informal theorems. Any general geometric proposition or problem may be considered an informal theorem. The following examples may be taken as instances of this sort.

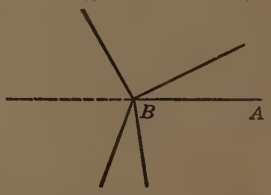
EXAMPLES

1. The sum of all the angles on one side of a straight line at a given point is two right angles.



Proof. In the adjacent figure, AMB is a straight line, and the angles a, c, n, r have a common vertex at M; hence their sum is a straight angle, or two right angles.

2. The sum of all the angles about a point is four right angles.



Proof. In the adjacent figure one side of any angle, as AB, can be extended through the common vertex. Then the sum of all the angles on one side of the line is a straight angle, and the sum on the other is also a straight angle. The total sum, then, is two straight angles, or four right angles.

The postulates in §§ 32 and 34 might also be regarded as informal theorems.

45. Hypothesis and conclusion. In the statement of every theorem there is a conditional part (often expressed by an "if" clause) called the *hypothesis.*

The other part of the theorem is a consequence of the hypothesis and is called the *conclusion.*

HYPOTHESIS	CONCLUSION
If equals are divided by equals,	the results are equal.
If it rains,	the air will be cooler.

Sometimes the condition, instead of being expressed by an "if" clause, is implied; for example:

HYPOTHESIS	CONCLUSION
Supplements of equal angles	are equal.
(If two angles are supplements of equal angles,	they are equal.)
Two perpendiculars to the same line	are parallel.
(If two lines are perpendicular to the same line,	they are parallel.)
The area of a rectangle	equals the product of its base and altitude.
(If a figure is a rectangle,	then its area equals the product of its base and altitude.)

In the theorems stated in the implied form it is usually easier to pick out the conclusion first, as that is the statement to be proved. The remainder of the theorem is the hypothesis.

EXERCISES

In the following, state the hypothesis and the conclusion. Where necessary, restate the implied conditions in the "if" clause form.

1. If the sun comes out, it will be warmer.

2. If two angles are supplements of the same angle, they are equal.

3. Complements of equal angles are equal.

4. Love me, love my dog.

5. The diagonals of a square bisect each other.

6. The more haste, the less speed.

7. The angles opposite two equal sides of a triangle are equal.

8. If the number of automobiles increases, the highways will improve.

9. If you try this car, you will buy it.

10. If airplanes continue to improve, transatlantic air travel will be established.

11. Parallel lines included between parallel lines are equal.

12. A quadrilateral having three right angles is a rectangle.

46. Nature of proof. A formal *proof*, or *demonstration*, is a connected piece of reasoning in which certain statements or assumptions which are either admitted to be true or have been proved beforehand are used to establish the truth of some other statement.

EXERCISES

In the following exercises a correct method of reasoning has been used. In which of them is a wrong assumption made? Which of the conclusions are incorrect?

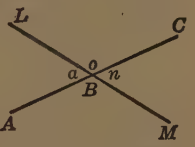
1. Newton lived before Washington. Washington lived before Lincoln. Therefore Newton lived before Lincoln.

2. William is the father of Robert. Robert is the father of Catherine. Therefore William is the grandfather of Catherine.

3. All men are created equal. Therefore all newborn babies look alike.

4. All birds are colored brown. A canary is a bird. Therefore the canary bird is brown.

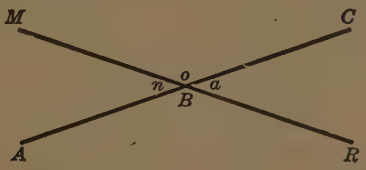
47. **Vertical angles.** Two angles are *vertical* if the sides of one are prolongations of the sides of the other.



In the adjacent figure, $\angle a$ and $\angle n$ are vertical angles; $\angle o$ and $\angle ABM$ are also vertical angles.

The following is an illustration of a geometric theorem proved formally:

48. *If two straight lines intersect, the vertical angles are equal.*



Given the straight lines AC and MR intersecting at B, forming the vertical angles a and n and also o and ABR.

To prove that $\angle a = \angle n$ and $\angle o = \angle ABR$.

Proof

STATEMENTS	REASONS
1. $\angle a + \angle o = 1$ straight angle.	1. By definition (§ 31).
2. $\angle n + \angle o = 1$ straight angle.	2. Same reason as for step 1.
3. Hence $\angle a + \angle o = \angle n + \angle o$.	3. All straight angles are equal (§ 32).
4. Then $\angle a = \angle n$.	4. Subtracting $\angle o$ from both members.
5. Similarly, $\angle o = \angle ABR$.	5. As in steps 1 to 4.

49. The parts of a proof. The details of a formal geometric proof, as may be seen in the demonstrations of §§ 21 and 48, fall under three general heads:

(*a*) A statement of the conditions in terms of the figure (the hypothesis).

(*b*) A statement of what is to be proved in terms of the figure (the conclusion).

(*c*) An ordered series of dependent steps, the first part of each being an assertion of some fact concerning the figure, and the second part the reason for the assertion. This reason may be a definition, an assumption (axiom or postulate), or some theorem already proved. As the demonstration progresses, each assertion is supported by some statement on which we are agreed. This affords a series of dependent steps, which proceed from the definitions, assumptions, or previous proofs to the truth (conclusion) which we set out to establish.

REVIEW QUESTIONS

1. In how many points can two straight lines intersect? two circles?

2. How many points are needed to fix the position of a straight line?

3. How many straight lines can be drawn through two points?

4. What is the difference in meaning between *line* and *line segment*?

5. What is a right angle? an acute angle? an obtuse angle? a straight angle?

6. What are adjacent angles? Illustrate.

7. What are complementary angles? Illustrate.

8. What are supplementary angles? Illustrate.

9. What are perpendicular lines?

10. If the angle between two lines is a right angle, are the lines perpendicular to each other?

11. Can two straight lines inclose a space?

12. What is an axiom? Illustrate.

13. What is a postulate? Illustrate.

14. What is a theorem? Illustrate.

15. State a theorem which has been proved informally.

16. State two theorems which have been proved formally.

17. Is the word *point* defined in the text? the word *line*?

18. What are vertical angles? Illustrate.

19. What is the hypothesis of a theorem? the conclusion?

20. Illustrate each.

21. What is the vertex of an angle?

22. How many right angles are there in a straight angle?

23. How many right angles can be placed around a point as a vertex?

24. How many straight angles can be placed around a point as a vertex?

25. How many supplements does a given angle have?

26. How many different pairs of angles may be complementary to each other?

27. If two angles are complementary to the same angle, are they necessarily equal to each other?

28. If two angles are supplementary to each other, are they necessarily equal to each other?

29. If $x°$ denotes an angle, how should you denote its complement? What does $(180 - x)°$ denote?

30. If $x°$ denotes an angle, what does $(90 - x)°$ denote?

31. What is the complement of the angle $(90 - x)°$?

32. If $x°$ denotes one angle, what denotes an angle 90° greater than $x°$?

33. If x is an acute angle, which is greater, its complement or its supplement? How much greater?

34. If the sum of two adjacent angles is a straight angle, do the exterior sides of these angles form a straight line? Draw a figure and illustrate.

Complete the following:

35. Two angles having a common side and the same vertex are ____.

36. If two straight lines intersect so that two adjacent angles are equal, the lines are ____.

37. The sum of all the successive angles on one side of a straight line, having a common vertex in the line, is ____.

38. Complements of equal angles are ____.

39. Supplements of equal angles are ____.

40. The supplement of an acute angle is ____ angle.

41. The supplement of a right angle is ____ angle.

42. The complement of an acute angle is ____ angle.

43. At a given point in a line one and only one ____ can be drawn.

Assumptions

50. Axioms. The following axioms, or arithmetic assumptions, were all used in algebra and are restated here for convenience of reference.

Axiom 1. Numbers equal to the same number are equal to each other.

Axiom 2. The whole is equal to the sum of all its parts.

Axiom 3. If equals are added to equals, the results are equal.

Axiom 4. If equals are subtracted from equals, the results are equal.

Axiom 5. If equals are multiplied by equals, the results are equal.

Axiom 6. If equals are divided by equals (not zero), the results are equal.

Axiom 7. A number may be substituted for its equal in any operation on numbers.

51. Postulates. Some of the following postulates or geometric assumptions have already been used; the others will be needed in the work which is to follow and are stated here for convenience of reference.

Postulate 1. Through two points one and only one straight line can be drawn.

Postulate 2. A straight line segment is the shortest line between two points.

Postulate 3. A straight line segment can be extended any distance.

Postulate 4. A straight line segment can be bisected (separated into two equal parts) by only one point.

Postulate 5. An angle can be bisected by one and only one line.

Postulate 6. *At a given point on a line there is one and only one line perpendicular to it.*

Postulate 7. *Any geometric figure may be moved from one place to another without changing its size or shape.*

Postulate 8. *All straight angles are equal.*

Postulate 9. *A circle may be drawn with any center and any radius.*

Postulate 10. *Through a given point outside a line there is one and only one line parallel to it.*

SYMBOLS AND ABBREVIATIONS

$+$	plus	adj.	adjacent
$-$	minus	alt.	alternate
$\times$, $\cdot$	times	ax.	axiom
$\div$, :	divided by	comp.	complement
$=$	equals, is equal to	const.	construction
$\neq$	is not equal to	cor.	corollary
$<$	is less than	corr.	corresponding
$>$	is greater than	def.	definition
$\therefore$	therefore	ex.	exercise
$\angle$	angle	ext.	exterior
$\angle$s	angles	hyp.	hypothesis
$\perp$	perpendicular, is perpendicular to	iden.	identical
		int.	interior
$\perp$s	perpendiculars	isos.	isosceles
$\parallel$	parallel, is parallel to	post.	postulate
$\sim$	similar, is similar to	prop.	proposition
$\cong$	is congruent with	rect.	rectangle
$\triangle$	triangle	rt.	right
$\triangle$s	triangles	sq.	square
$\square$	parallelogram	st.	straight
$\square$	rectangle	a', a'', a''', $\cdots$	a-prime, a-second, a-third, etc.
$\odot$	circle		
$\odot$s	circles	a_1, a_2, a_3, $\cdots$	a-one, a-two, a-three, etc.
$\cdots$	and so on		

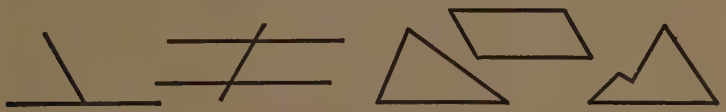
BOOK I

STRAIGHT-LINE FIGURES

What science can there be more noble, more excellent, more useful for men, more admirably high and demonstrative, than this of the mathematics? — BENJAMIN FRANKLIN

52. Rectilinear figures. A *rectilinear* figure is one composed only of straight lines.

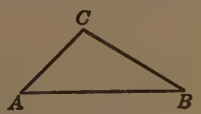
We consider here only those figures all of whose parts lie in one flat surface, or plane.



Rectilinear figures

53. Triangle. The figure formed by three line segments joining three points which are not in a straight line is called a *triangle*.

In the triangle ABC the points A, B, and C are called the vertices, the line segments AB, BC, and CA are called the sides, and the angles ABC, BCA, and CAB are called angles of the triangle.



It is to be noted that, by definition, a triangle does not include as one of its parts the portion of the plane bounded by the lines AB, BC, and CA. In other words, the triangle is not the surface bounded by the straight lines AB, BC, and CA, but the figure formed by the lines.

54. Triangles classified by sides. A triangle is called *isosceles* if two of its sides are equal.

55. A triangle is called *equilateral* if all its sides are equal.

Isosceles triangle Isosceles triangle Equilateral triangle

56. Triangles classified by angles. A triangle is called a *right* triangle if one of its angles is a right angle; an *obtuse* triangle if one angle is an obtuse angle; an *acute* triangle if all its angles are acute angles; an *equiangular* triangle if all its angles are equal.

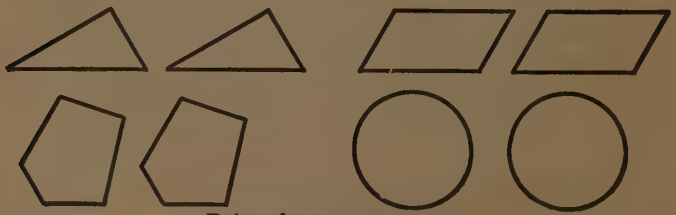
57. Hypotenuse. The *hypotenuse* of a right triangle is the side opposite the right angle.

Right triangle Obtuse triangle Acute triangle Equiangular triangle

58. Parts of a triangle. Every triangle has six parts: three sides and three angles.

59. Congruent figures. Two figures which have precisely the same size and shape are called *congruent* (cŏn'grū ĕnt) figures.

The following pairs of figures are congruent:



Pairs of congruent figures

Congruent figures are alike in all their parts. Thus, if two triangles are congruent, each angle and each side of one has a part just like it in the other.

60. Congruent triangles. Since a triangle has six parts, three sides and three angles, a very important question concerning two triangles is: What parts of one triangle must be equal to the corresponding parts of another triangle in order that the triangles may be congruent?

To answer this question it is necessary to imagine figures of the triangles drawn on paper to be moved about, or even turned completely over, and one placed upon the other. If we are cutting two pieces of paper, or metal, or board to the same pattern, we would naturally place one on the other to see if the two fit; that is, to see if they are the same size and shape, or are congruent. Since in the study of geometry we do not deal with cardboard or wire triangles but with triangles made up of geometric lines, this process of superposing, as it is called, is usually an imaginary one. Nevertheless it is very helpful in proving that under certain conditions two triangles are congruent.

Two geometric figures, then, are congruent if they can be made to *coincide*.

61. The superposition postulate. In § 51 the following postulate of free motion, as it is called, was stated: Any geometric figure may be moved from one place to another without changing its size or shape.

In order that we may find out whether certain figures are congruent or not we must assume the preceding postulate.

QUERY 1. Are two straight lines always congruent?

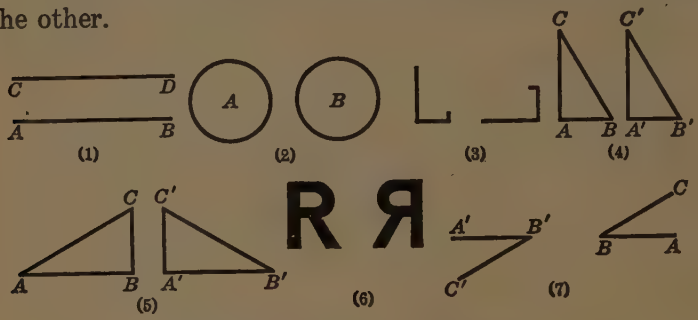
QUERY 2. If two circles have equal radii, are they congruent?

QUERY 3. Are two angles that contain the same number of degrees congruent?

QUERY 4. May two angles be congruent if the sides are not the same length?

EXERCISE

The following pairs of figures are congruent. Describe the motion by which one may be made to coincide with the other.



Introduction to Theorem 1

EXERCISES

1. Using the ruler and the protractor, draw on separate sheets of paper two triangles having one angle 50° and the sides including the angle 3 in. and 4 in. respectively. Superpose one triangle upon the other. Do they coincide?

2. Repeat Exercise 1 with an angle of 75° and sides 8 centimeters and 10 centimeters respectively.

3. Repeat Exercise 1 when two sides are 9 centimeters and 7 centimeters respectively and their included angle contains 110°.

4. What do Exercises 1–3 indicate regarding two triangles which have two sides and the included angle of one equal respectively to two sides and the included angle of the other?

5. Complete the statement: If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, they are _____.

Theorem 1

62. *If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another, the two triangles are congruent.*

Given $\triangle ABC$ **and** $\triangle A'B'C'$, **in which** $c = c'$, $b = b'$, **and** $\angle A = \angle A'$.

To prove that $\triangle ABC \cong \triangle A'B'C'$.

Method. Place one triangle upon the other so that one pair of equal sides coincide, and at the same time make one of the equal angles fall upon the other. Then prove that the remaining parts of the two triangles coincide.

Proof

STATEMENTS	REASONS
1. Place $\triangle ABC$ upon $\triangle A'B'C'$ in such a way that c coincides with its equal, c', and so that point A falls on A' and B on B'.	1. § 51, Postulate 7. Any geometric figure may be moved from one place to another without changing its size or shape.
2. b will fall on b'.	2. $\angle A$ is given equal to $\angle A'$.
3. C will fall on C'.	3. b is given equal to b'.
4. $\therefore$ since B falls on B' and C on C', a will coincide with a'.	4. § 51, Postulate 1. Through two points one and only one straight line can be drawn.
5. $\triangle ABC \cong \triangle A'B'C'$.	5. The $\triangle$ coincide.

NOTE. When referred to later this theorem will often be abbreviated thus: s.a.s. = s.a.s.

QUERY 1. Can $\angle A$ be placed on $\angle A'$ so that the triangles do not coincide? How?

QUERY 2. Where would AC fall if the triangles were superposed as in the proof and $\angle A$ were greater than $\angle A'$?

QUERY 3. Can $\triangle ABC$ be placed on $\triangle A'B'C'$ so that AB coincides with $A'B'$ and yet the triangles do not coincide? If so, how?

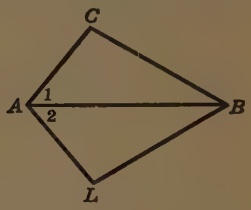
QUERY 4. What is the hypothesis in Theorem 1? the conclusion?

QUERY 5. What postulate is necessary in order to begin a proof by superposition?

HOW A THEOREM IS USED

63. Application of Theorem 1. The essential part one theorem plays in proving another is illustrated in the following demonstration.

ILLUSTRATIVE DEMONSTRATION



Given the figure above, in which $AC = AL$ and AB bisects $\angle CAL$.

***To prove that* $\triangle ABC \cong \triangle ABL$.**

Proof

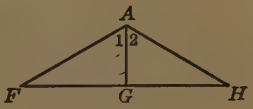
STATEMENTS	REASONS
1. $AC = AL$.	1. By hypothesis.
2. $AB = AB$.	2. Identical.
3. $\angle 1 = \angle 2$.	3. By hypothesis.
4. $\therefore \triangle ABC \cong \triangle ABL$.	4. If two ▲ have two sides and the included ∠ of one equal respectively to two sides and the included ∠ of the other, the ▲ are congruent.

It would be a waste of time here to prove again that the two triangles are congruent by superposition and work out the proof as in Theorem 1. The better way is to show that the same hypothesis exists here as exists in Theorem 1 (two sides and the included angle of one equal etc.). Then the same conclusion that follows in Theorem 1 must follow here; that is, the triangles are congruent. The following exercises give practice of this kind with Theorem 1.

EXERCISES ON THEOREM 1

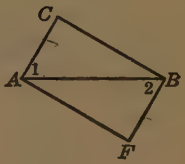
1. Given $\triangle FAH$, in which $AF = AH$ and AG bisects $\angle FAH$.

Prove $\triangle FGA \cong \triangle HGA$.



2. Given the figure $AFBC$, in which $AC = BF$ and $\angle 1 = \angle 2$.

Prove $\triangle ABC \cong \triangle ABF$.

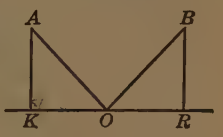


3. Given $\triangle ABL$, in which AH is $\perp$ to BL and $HB = HL$.

Prove $\triangle AHB \cong \triangle AHL$.

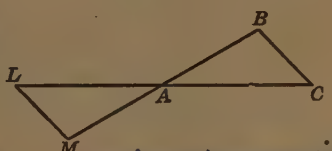
4. Given the adjacent figure, in which O is the mid-point of KR, $AK = BR$, and each is perpendicular to KR.

Prove $\triangle AKO \cong \triangle BRO$.



5. Given the figure $LMCB$, in which $AL = AC$ and $AM = AB$.

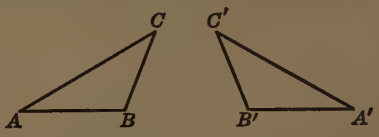
Prove $\triangle AML \cong \triangle ABC$.



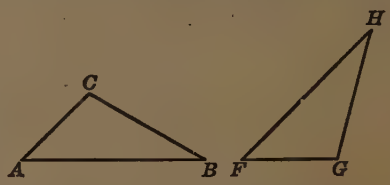
vertically opposite

64. Corresponding parts of congruent triangles. In congruent triangles *corresponding* angles lie opposite the equal sides and *corresponding* sides lie opposite the equal angles.

Corresponding parts of congruent triangles are necessarily equal, since they coincide if the triangles are made to coincide.



Thus, if $\triangle ABC$ and $\triangle A'B'C'$ are congruent and if in these triangles $\angle A = \angle A'$, $\angle B = \angle B'$, and $\angle C = \angle C'$, then the sides opposite each pair of equal angles are corresponding and therefore equal; that is, $BC = B'C'$, $AC = A'C'$, and $AB = A'B'$.



Similarly, if $\triangle ABC$ and $\triangle FGH$ are congruent and if $AB = FH$, $BC = GH$, and $CA = FG$, then the corresponding angles lie opposite the equal sides and are equal to each other; that is, $\angle C = \angle G$, $\angle A = \angle F$, and $\angle B = \angle H$.

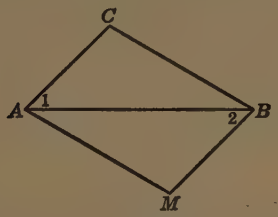
65. Fundamental method. The method usually employed in geometry for proving two line segments of a given figure equal or two of its angles equal is to show that *they are corresponding parts of congruent triangles.* There are two steps in the process:

(*a*) Select two triangles which contain the lines or angles which are to be proved equal and show that they are congruent triangles.

(*b*) Show that the parts to be proved equal are corresponding parts of these two congruent triangles.

The demonstration which follows illustrates this important method.

ILLUSTRATIVE DEMONSTRATION



Given the figure above, in which $AC = BM$ and $\angle 1 = \angle 2$.

To prove that $BC = AM$ and $\angle C = \angle M$.

Proof

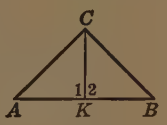
STATEMENTS	REASONS
1. $AC = BM$.	1. By hypothesis.
2. $AB = AB$.	2. Identical.
3. $\angle 1 = \angle 2$.	3. By hypothesis.
4. $\therefore \triangle ABC \cong \triangle ABM$.	4. § 62. s.a.s. = s.a.s.
5. $\therefore BC = AM$ and $\angle C = \angle M$.	5. § 64. Corresponding parts of congruent △ are equal.

EXERCISES

In the following exercises show that the conclusion follows from the hypothesis:

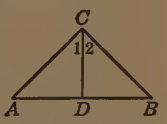
1. Hypothesis: In $\triangle ABC$ line KC bisects AB and is perpendicular to it.

Conclusion: $AC = BC$.



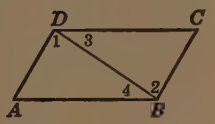
2. Hypothesis: In $\triangle ABC$ line $AC = CB$ and CD bisects $\angle ACB$.

Conclusion: $\angle A = \angle B$.

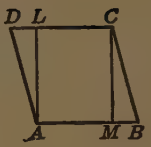


3. Hypothesis: In figure $ABCD$ line $AD = BC$ and $\angle 1 = \angle 2$.

Conclusion: $AB = DC$, $\angle A = \angle C$, and $\angle 3 = \angle 4$.

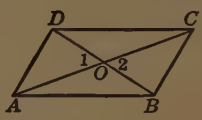


4. Hypothesis: In the figure $ABCD$ side $AB = BC = CD = DA$, and $\angle D = \angle B$ and $LC = AM$.



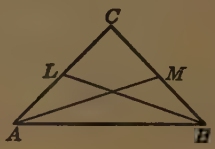
Conclusion: $AL = CM$.

5. Hypothesis: AC and BD are bisected at O.



Prove $\triangle AOD \cong \triangle BOC$ and $AD = BC$.

6. The figure ABC is a triangle in which $AC = BC$. Line AM is drawn to the mid-point of BC and BL to the mid-point of AC.



Prove $\triangle ACM$ and BCL congruent and $AM = BL$.

Introduction to Theorem 2

EXERCISES

1. Using ruler and protractor, draw two triangles which have one side 3.5 in. and the angles adjacent to that side 42° and 56° respectively. Then place one triangle on the other, making them coincide if possible.

2. Repeat the work of Exercise 1, making the side 10 centimeters and the angles 70° and 30°.

3. Using a side equal to 4.2 in. and the two angles 46° and 97° respectively, repeat the work of Exercise 1.

4. What conclusion seems probable concerning the congruence of two triangles in which a side and two adjacent angles of one are equal respectively to a side and two adjacent angles of the other? Does it make any difference in the result if the two angles are interchanged in one of the triangles?

Theorem 2

66. *If a side and the two adjacent angles of one triangle are equal respectively to a side and the two adjacent angles of another, the triangles are congruent.*

Given $\triangle ABC$ **and** $A'B'C'$, **in which** $c = c'$, $\angle A = \angle A'$, **and** $\angle B = \angle B'$.

To prove that $\triangle ABC \cong \triangle A'B'C'$.

Method. Superposition. Place one triangle upon the other so that the equal sides and the equal angles coincide. Then prove that the other parts also coincide. The triangles are then congruent.

Proof

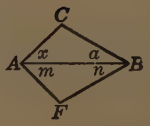
STATEMENTS	REASONS
1. Superpose $\triangle A'B'C'$ on $\triangle ABC$ so that c' falls on its equal, c, the point A' falling on the point A, and the point B' on the point B.	1. § 51, Postulate 7. Any geometric figure may be moved from one place to another without changing its size or shape.
2. b' will fall on b.	2. By hypothesis, $\angle A = \angle A'$.
3. a' will fall on a.	3. By hypothesis, $\angle B = \angle B'$.
4. The point C' falls at the intersection of the sides b and a.	4. It falls on both the line AC and the line BC.
5. $\therefore \triangle ABC \cong \triangle A'B'C'$.	5. They coincide in all their parts.

NOTE. When referred to later this proposition will often be abbreviated thus: a. s. a. = a. s. a.

EXERCISES ON THEOREM 2

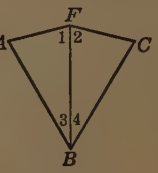
1. In the adjacent figure, $\angle x = \angle m$ and $\angle a = \angle n$.

Prove $\triangle ABC \cong \triangle ABF$ and $AC = AF$.



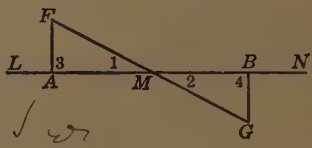
2. In the adjacent figure, BF bisects $\angle ABC$ and $\angle AFC$.

Prove $\triangle ABF \cong \triangle BCF$ and $\angle A = \angle C$.



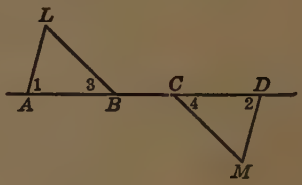
3. In the adjacent figure, M is the mid-point of AB with FA and GB each $\perp$ to LN.

Prove $MF = MG$ and $\angle F = \angle G$.



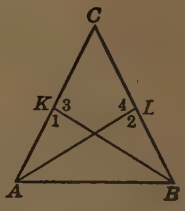
4. In the adjacent figure, $AC = DB$, $\angle 1 = \angle 2$, and $\angle 3 = \angle 4$.

Prove $AL = DM$ and $\angle L = \angle M$.



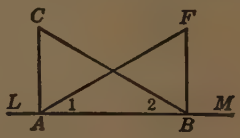
5. In the adjacent figure, $\angle CAB = \angle CBA$, AL bisects $\angle CAB$, and BK bisects $\angle CBA$.

Prove $\triangle ABL \cong \triangle ABK$; and $AK = BL$, $\angle 1 = \angle 2$, and $\angle 3 = \angle 4$.



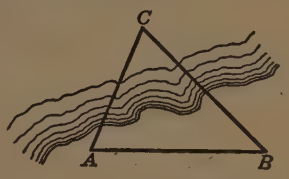
6. Hypothesis: In the adjacent figure, AC and BF are each perpendicular to LM and $\angle 1 = \angle 2$.

Conclusion: $AF = BC$, $\angle C = \angle F$, and $AC = BF$.



Use of Theorem 2 in Navigation and Aviation

Theorem 2 has been used to determine by radio the position of a ship at sea in foggy weather. The operator on the ship and the operators at two stations, A and B, on land must be able to send and receive messages, and the latitude and the longitude of these stations must be known and also their distance apart. Signals from C enable the operator at A to determine the angle CAB and the operator at B to determine the angle CBA. With these two angles and the distance AB the operator at either A or B can quickly compute the distances AC and BC and the latitude and longitude of point C. The result may then be sent to C.



The computation referred to may be made in either one of two ways. The more exact method would make use of trigonometry, by means of which one can find the remaining parts of a triangle if a side and the two adjacent angles are known. The other method merely involves drawing a triangle in which the number of miles in AB is represented by any convenient number of inches and in which the adjacent angles are equal to A and B.

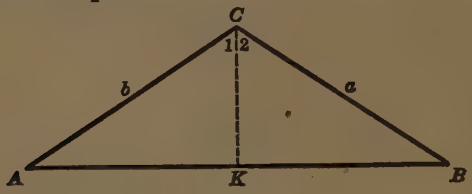
The same method enables the aviator to determine his position even when enveloped in fog. This was done in one of the early flights from New York to France, in which the aviator was over the open ocean for nineteen hours in a fog, but was able to determine his position by wireless communication with ships at sea which were also in fog.

Query 1. What is an isosceles triangle?
Query 2. What did Exercises 16–19 of page 11 indicate regarding two angles of an isosceles triangle?

compute: Trigonometry

Theorem 3

67. ***If a triangle is isosceles, the angles opposite the equal sides are equal.***



Given $\triangle ABC$, **in which** $a = b$.

To prove that $\angle A = \angle B$.

Method. Bisect the vertex angle, and prove that the two triangles thus formed are congruent. Then prove that the base angles are corresponding parts of the two congruent triangles.

Proof

STATEMENTS	REASONS
1. Let CK bisect $\angle ACB$ and meet AB at some point K.	1. § 51, Postulate 5. An $\angle$ can be bisected by one and only one line.
2. Then, in $\triangle ACK$ and $\triangle BCK$, $\angle 1 = \angle 2$.	2. By construction.
3. $AC = BC$.	3. By hypothesis.
4. $CK = CK$.	4. Identical.
5. $\therefore \triangle ACK \cong \triangle BCK$.	5. § 62. If two sides and the included $\angle$ of one $\triangle$ are equal respectively to two sides and the included $\angle$ of another, the two $\triangle$ are congruent.
6. $\angle A = \angle B$.	6. § 64. Corresponding parts of congruent $\triangle$ are equal.

68. Corollary. A *corollary* is a secondary or a minor theorem dependent on a major theorem. Usually its truth is apparent from the major theorem or follows from it by means of a short proof.

69. ***Corollary 1.*** *The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to it.*

HINT. In the figure for Theorem 3 show that $\triangle CKA \cong \triangle CKB$. Then $AK = KB$, $\angle AKC = \angle BKC$, and CK is $\perp$ to AB (§ 35).

70. ***Corollary 2.*** *An equilateral triangle is also equiangular.*

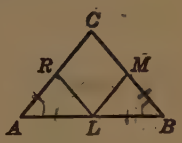
HINT. $a = b$, hence $\angle B = \angle A$. Also, $a = c$, hence $\angle C = \angle A$. Etc.

EXERCISES ON THEOREMS 1, 2, AND 3

1. Draw a figure for Corollary 1, state the hypothesis and the conclusion, and write out a proof.

2. Proceed as in Exercise 1 with Corollary 2.

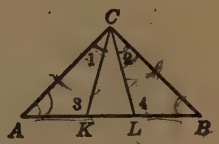
3. In the adjacent figure, $AC = BC$. Points L, M, and R are the mid-points of AB, BC, and AC respectively.



Prove $LM = LR$.

HINT. Prove $\triangle ALR \cong \triangle BLM$.

4. In the adjacent figure, $AC = BC$ and $\angle 1 = \angle 2$.



Prove $CK = CL$ and $\angle 3 = \angle 4$.

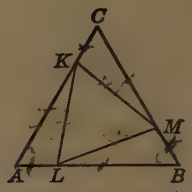
5. In the adjacent figure, $CL = CM$, and $AL = BM$.

Prove $AC = BC$.

HINT. Prove $\angle 1 = \angle 2$.

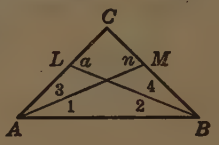
6. In the adjacent figure, ABC is an equilateral triangle and $AL = BM = CK$.

Prove that $\triangle KLM$ is equilateral.



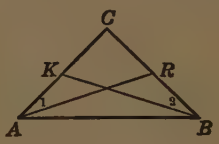
7. In the adjacent figure, $AC = BC$, and AM and BL bisect the base angles.

Prove $\triangle CAM \cong \triangle CBL$, $CL = CM$, and $\angle a = \angle n$.



8. In the adjacent figure, $AC = BC$, and K is the mid-point of AC, while R is the mid-point of BC.

Prove $\triangle CAR \cong \triangle CBK$, $AR = BK$, and $\angle 1 = \angle 2$.



9. If the base of an isosceles triangle is divided into three equal parts, prove that the lines joining the points of division with the vertex are equal.

10. The bisectors of the base angles A and B of the isosceles triangle ABC meet the sides AC and BC at K and R. Prove $\triangle ABK \cong \triangle ABR$, $AR = BK$, and $AK = BR$.

Introduction to Theorem 4

Exercises

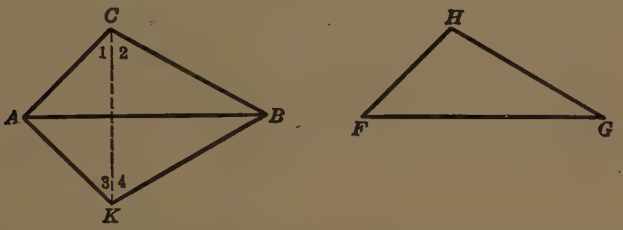
1. Construct two triangles each of which has sides 5 centimeters, 6 centimeters, and 7 centimeters. Place one upon the other and make them coincide if possible.

2. Repeat Exercise 1 for two triangles having sides of 3.5 in., 4 in., and 4.5 in. respectively.

3. What conclusion from Exercises 1 and 2 appears probable as to the congruence of any two triangles which have three sides equal respectively each to each?

Theorem 4

71. *If the three sides of one triangle are equal respectively to the three sides of another, the triangles are congruent.*



Given $\triangle ABC$ **and** $\triangle FGH$, **in which** $AB = FG$, $BC = GH$, **and** $AC = FH$.

To prove that $\triangle ABC \cong \triangle FGH$.

Method. Place $\triangle FGH$ in the position of $\triangle ABK$, draw CK, and prove that $\angle 1 + \angle 2$, or $\angle C$, is equal to $\angle 3 + \angle 4$, or $\angle K$, which equals $\angle H$. Then two sides and the included angle of one triangle equal respectively two sides and the included angle of the other.

Proof

Let AB and FG be the longest sides of the triangles respectively.

Statements	Reasons
1. Suppose $\triangle FGH$ is placed adjacent to $\triangle ABC$ so that FG coincides with its equal, AB, the point F falling on the point A, the point G on the point B, and the point H at K.	1. § 51, Postulate 7. Any figure may be moved from one place to another without changing its size or shape.
2. Draw CK, forming $\angle 1$ and $\angle 2$ at C and $\angle 3$ and $\angle 4$ at K.	2. § 51, Postulate 1. Through two points one and only one straight line can be drawn.

3. $AC = AK$.	3. By hypothesis.
4. $\therefore \angle 1 = \angle 3$.	4. § 67. If a $\triangle$ is isosceles, the $\measuredangle$ opposite the equal sides are equal.
5. Also, $BC = BK$.	5. By hypothesis.
6. $\therefore \angle 2 = \angle 4$.	6. § 67.
7. $\angle 1 + \angle 2 = \angle 3 + \angle 4$, or $\angle ACB = \angle AKB$.	7. § 50, Axiom 3. If equals are added to equals, the results are equal.
8. $\triangle ABC \cong \triangle ABK$.	8. § 62. If two sides and the included $\angle$ of one $\triangle$ are equal respectively to two sides and the included $\angle$ of another, the two $\triangle$ are congruent.
9. Then $\triangle ABC \cong \triangle FGH$.	9. Two figures congruent to a third are congruent to each other.

NOTE. When referred to later this theorem will often be abbreviated thus: S.S.S. = S.S.S.

QUERY 1. Does the proof of Theorem 4 depend on Theorem 3? on Theorem 1? If the proof of a theorem depends on Theorem 3, does it necessarily depend on Theorem 1?

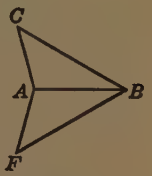
QUERY 2. If three angles of one triangle are equal respectively to three angles of another, are the triangles congruent? Illustrate.

QUERY 3. Draw a figure for Theorem 4, assuming AB and FG the shortest sides respectively of the triangles. Would any difficulty then arise in the proof?

QUERY 4. What difficulty arises if you try to prove Theorem 4 by superposition?

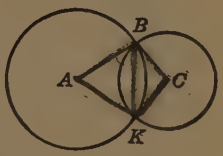
72. Auxiliary lines. Lines other than those named or implied in the hypothesis of the theorem are often an aid or a necessity in order to provide congruent triangles which contain the lines or angles to be proved equal. Such lines are called *auxiliary lines*. They should be drawn whenever they are seen to be necessary.

EXERCISES ON THEOREMS 1, 2, 3, AND 4



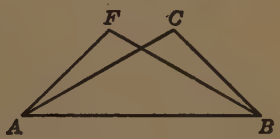
1. In the figure, $AC = AF$ and $BC = BF$.

Prove $\triangle ABC \cong \triangle ABF$ and $\angle C = \angle F$.



2. Two circles, whose centers are A and C, intersect at B and K.

Prove $\angle AKC = \angle ABC$.

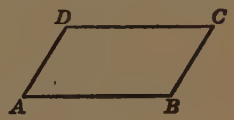


3. In the figure, $AF = BC$ and $AC = BF$.

Prove $\angle F = \angle C$.

4. In an equilateral $\triangle ABC$, K and L are the mid-points of sides AB and BC respectively.

Prove $AL = CK$.

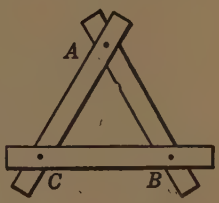


5. The opposite sides of the quadrilateral $ABCD$ are equal.

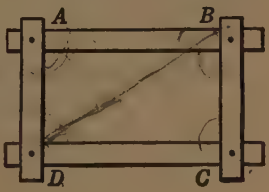
Prove $\angle B = \angle D$ and $\angle A = \angle C$.

HINT. Apply Theorem 4 and § 72.

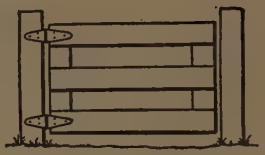
QUERY 1. Will three bars held together by one bolt at A, B, and C respectively form a rigid figure?



QUERY 2. Will a quadrilateral formed by four bars and held together by one bolt at A, B, C, and D respectively form a rigid figure? How can it be made rigid?



QUERY 3. Will a gate (like the adjacent figure) tend to sag from its own weight? If so, how can sagging be prevented?



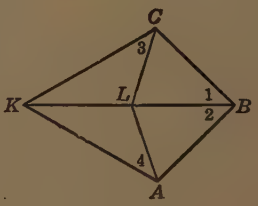
73. Auxiliary triangles. Sometimes the hypothesis of a theorem is such that the required lines or angles cannot be proved equal by showing directly that they are corresponding parts of two congruent triangles. In such theorems, however, the hypothesis usually makes it possible to prove another pair of triangles congruent. Then the corresponding parts of these triangles may be used to prove the required pair congruent. The usefulness and necessity of this procedure is illustrated by the following exercises, which will offer some practice in selecting the proper pairs of *auxiliary* triangles.

EXERCISES

1. In the adjacent figure, $AB = BC$, $\angle 1 = \angle 2$, and L is any point on KB.

Prove $\angle 3 = \angle 4$.

HINT. First prove $\triangle AKB$ and BCK congruent. Then prove $\triangle AKL$ and CKL congruent.

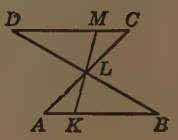


2. The opposite sides of $ABCD$ are equal, and MK is drawn through L, the mid-point of BD.

Prove $KL = LM$.

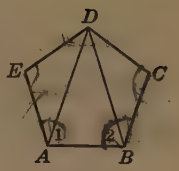
3. Lines AC and BD bisect each other at L, and a line through L cuts AB at K and DC at M.

Prove $KL = LM$.



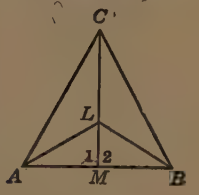
4. The figure $ABCDE$ has equal angles and equal sides.

Prove $\angle 1 = \angle 2$.



5. $AC = BC$ and $AL = BL$, and CL intersects AB at M.

Prove $\angle 1 = \angle 2$ and $AM = MB$.



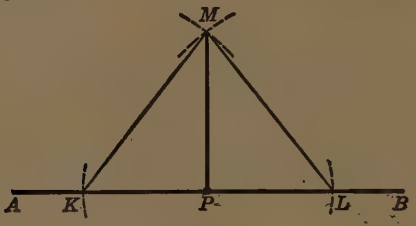
CONSTRUCTIONS

74. Constructions. The straight line and the circle are the only lines used in elementary geometry. Hence the only instruments needed to construct the figures which may be required are the straightedge and the compasses. In practical drawing other instruments besides these two are used, such as triangles, the T-square, the dividers, and the protractor. In mechanical drawing, however, the instruments are selected so that the required figures can be drawn rapidly and accurately. The construction (drawing) work of geometry limits itself to the use of the ruler and compasses, not only because with them alone the figures required can be drawn theoretically exact, but because the ruler and compasses are the simplest instruments with which all the constructions can be made.

In mechanical drawing a perpendicular at a point lying in a line or a perpendicular from a point to a line is drawn very easily by means of a drawing board, a T-square, and a triangle. These two problems are also solved in a simple manner by using only a straightedge and the compasses, as in Constructions 1 and 2 which follow.

Construction 1

75. *At a given point in a line construct a perpendicular to the line.*



Given P, any point on line AB.

Required to construct a perpendicular to AB ***at*** P.

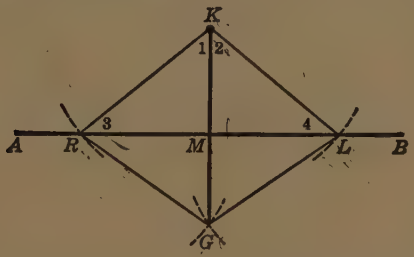
Method. With P as center, draw arcs intersecting AB at K and L. With K and L as centers and a radius greater than PK, draw two arcs intersecting at M. Draw MP. Then prove that MP is $\perp$ to AB at P.

Proof

1. Draw KM and LM.	1. § 51, Postulate 1. Through two points one and only one straight line can be drawn.
2. $KP = PL$.	2. By construction.
3. $KM = LM$.	3. By construction.
4. $MP = MP$.	4. Identical.
5. $\therefore \triangle MPK \cong \triangle MPL$.	5. § 71. s.s.s. = s.s.s.
6. $\angle MPK = \angle MPL$.	6. § 64. Corresponding parts of congruent triangles are equal.
7. $\therefore MP$ is $\perp$ to AB.	7. § 35. If one straight line meets another so as to make the two adjacent $\angle$s equal, the two lines are $\perp$ to each other.

Construction 2

76. *From a given point outside a line construct a perpendicular to the line.*



Given the line *AB* and the point *K* outside the line.

Required to construct a perpendicular to AB from K.

Method. With K as the center and a radius great enough to cut AB, draw arcs cutting AB at points R and L. Then with R and L as centers and any radius greater than half of RL, draw two arcs intersecting at G. Draw KG. Then prove that KG is $\perp$ to RL.

Proof

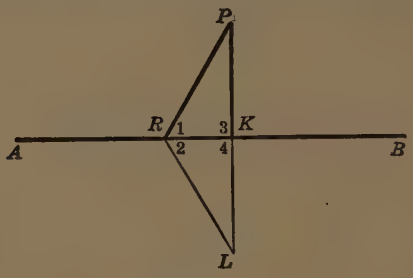
1. In $\triangle KGR$ and $\triangle KGL$, $KR = KL$ and $GR = GL$.	1. By construction.
2. $KG = KG$.	2. Identical.
3. $\therefore \triangle KGR \cong \triangle KGL$.	3. § 71. s.s.s. = s.s.s.
4. $\therefore \angle 1 = \angle 2$.	4. § 64.
5. In $\triangle KRL$, $KR = KL$.	5. By construction.
6. $\therefore KM$ is $\perp$ to RL and AB.	6. § 69.

77. ***Corollary.*** *If two points are equally distant from the extremities of a line, they determine its perpendicular bisector.*

HINT. Let K and G be the two points and RL the line. Then the proof is identical with the proof of Construction 2.

Theorem 5

78. *There is only one perpendicular from a point to a line.*



Given the point P any point outside the line AB, $PK \perp$ to AB at K, and PR any other line from P terminating in AB.

To prove that PR is not $\perp$ to AB.

Method. Prove that $\angle 1$ is not a right angle, as follows: Extend PK its own length to L. Draw LR, and prove that $\triangle PKR \cong \triangle LKR$ by s.a.s. = s.a.s. Then show that $\angle 1 = \angle 2$ and that $\angle 1 + \angle 2$ is not a straight angle and hence $\angle 1$ is not a right angle.

Proof

1. Let PK be extended to L, making $KL = PK$, and draw LR.	1. § 51, Postulate 3. A straight line segment can be extended any distance.
2. Now in $\triangle PKR$ and LKR, $\angle 3 = \angle 4$.	2. § 36. All rt. $\angle$s are equal.
3. $PK = KL$.	3. By construction.
4. $KR = KR$.	4. Identical.
5. $\therefore \triangle PKR \cong \triangle LKR$.	5. § 62. s.a.s. = s.a.s.
6. $\angle 1 = \angle 2$.	6. § 64. Corresponding parts of congruent figures are equal.

7. Since PKL is a straight line, PRL is not a straight line.	7. § 51, Postulate 1. Through two points one and only one straight line can be drawn.
8. Hence $\angle PRL$ is not a st. $\angle$, and half of it is not a rt. $\angle$.	8. § 31. A st. angle is an $\angle$ whose sides extend in opposite directions from the vertex and form a straight line.
9. $\therefore \angle 1$ is not a rt. $\angle$.	9. § 36. A right $\angle$ is half a straight $\angle$.
10. $\therefore PR$ is not $\perp$ to AB.	10. For if PR were $\perp$ to AB, PRK would be a right angle. (§ 36. A right $\angle$ is an $\angle$ whose sides are $\perp$.)

79. Distance from a point to a line. The *distance from a point to a line* is the length of the perpendicular from the point to the line. The fact that there is only one such perpendicular justifies here the use of the definite article "the" instead of the indefinite article "a" in defining distance from a point to a line.

80. Altitude of a triangle. An *altitude* of a triangle is a perpendicular from any vertex to the side opposite.

A *median* of a triangle is a line from any vertex to the mid-point of the opposite side.

QUERY 1. How many altitudes has a triangle?
QUERY 2. In what kind of triangles is a side also an altitude?
QUERY 3. Can all three sides of a triangle be altitudes?
QUERY 4. Can two sides of a triangle each be an altitude?

EXERCISES

1. At points L and M of line AB construct perpendiculars to AB on opposite sides of it.

2. Construct an equilateral triangle having sides about 3 in. long. From each vertex construct a perpendicular to the opposite side. How do these compare in length?

3. Construct a triangle whose sides are 3 in., 4 in., and 5 in. respectively. Then construct the altitude to the side 5.

4. Construct an isosceles triangle whose equal sides are each 3.5 in. Then construct its three altitudes. Compare their lengths.

5. Construct a triangle with three unequal sides and then construct its three altitudes. Measure the sides and the altitudes upon each. Is the longest altitude perpendicular to the longest side? If not, what is the case?

6. Draw a triangle with one angle greater than a right angle. Then construct the altitude to the shortest side.

7. Construct a right triangle in which the hypotenuse is twice the shorter side.

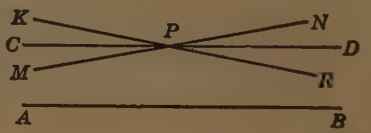
8. Prove that two of the three angles of a triangle cannot be right angles.

81. Parallel lines. *Parallel* lines are lines that lie in the same plane and do not meet however far they are produced.

Parallel lines

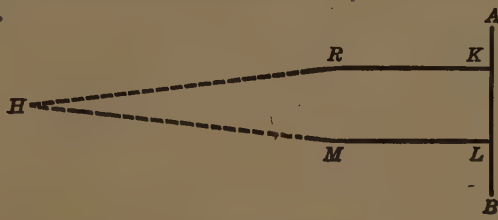
82. The postulate of parallels. Through a given point outside a line there can be one parallel to it and only one.

Thus, if CD passes through P and is parallel to AB, no other line, such as KR or MN, can pass through P and be parallel to AB.



Theorem 6

83. *Two lines perpendicular to the same line are parallel.*



Given the lines *KR* and *LM*, each ⊥ to *AB* at *K* and *L* respectively.

To prove that KR is ∥ to LM.

Method. Suppose that *KR* and *LM* intersect at *H*. Then there will be two perpendiculars to *AB* from *H*. But this is impossible, by Theorem 5.

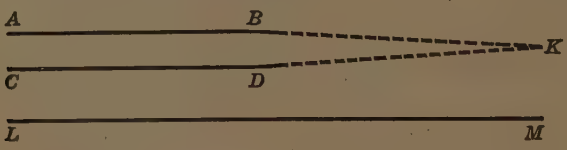
Proof

1. Since *KR* and *LM* lie in the same plane, they either meet or they do not meet.	1. No other possibility exists.
2. If they meet, as at *H*, there will be two ⊥s from that point to *AB*.	2. By hypothesis.
3. But this contradicts Theorem 5.	3. § 78. There is only one ⊥ from a point to a line.
4. Hence *KR* cannot meet *LM*.	4. Remaining alternative from step 1.
5. ∴ *KR* is ∥ to *LM*.	5. § 81.

In Theorem 6 the type of proof is called *indirect*. The method consists in assuming that the theorem is not true and then proving that this leads to an impossible result. Thus Theorem 6 asserts that *KR* and *ML* are parallel. The proof assumes that the two lines are not parallel and really intersect. The demonstration shows that the two lines cannot intersect. Hence the lines are parallel.

Theorem 7

84. *If two lines are parallel to a third line, the two lines are parallel to each other.*



Given the lines AB, CD, and LM in the same plane, with AB and $CD \parallel$ to LM.

To prove that AB is $\parallel$ to CD.

Method. Show that if the two lines meet, it will contradict the postulate of parallels. Hence the two cannot meet and are parallel.

Proof

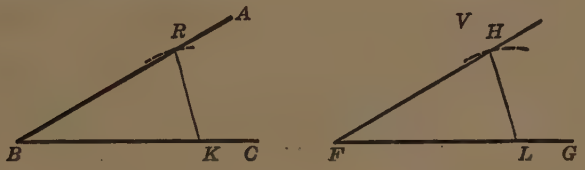
1. The lines AB and CD either meet or do not meet.	1. No other possibility exists.
2. If they meet, as at K, there will be two lines through $K \parallel$ to LM.	2. By hypothesis.
3. But this contradicts the postulate of parallels.	3. § 82. Through a given point outside a line there can be one parallel to it and only one.
4. AB and CD cannot meet.	4. Remaining alternative from step 1.
5. $\therefore AB$ is $\parallel$ to CD.	5. § 81.

QUERY 1. Are all the lines that can be drawn perpendicular to a given line necessarily parallel to each other?

QUERY 2. If three lines are each parallel to the same line, are they parallel to each other?

Construction 3

85. *Given one side and the vertex, construct an angle equal to a given angle.*



Given one side FG, the vertex F, and the $\angle ABC$.

Required to construct an angle equal to $\angle ABC$ with F as the vertex and FG as one side.

Method. With B as the center and any radius, draw an arc cutting AB at R and BC at K. With BK as the radius and F as the center, draw arc VL cutting FG at L. With KR as the radius and L as the center, describe an arc cutting arc VL at H. Draw FH. Then $\angle F = \angle B$.

Proof

1. $BK = BR = FL = FH$, and $RK = HL$.	1. By construction.
2. $\therefore \triangle BKR \cong \triangle FLH$.	2. § 71. s. s. s. = s. s. s.
3. $\therefore \angle B = \angle F$.	3. § 64. Corresponding parts of congruent triangles are equal.

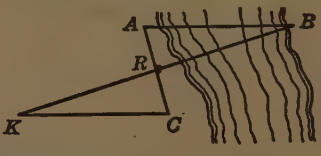
EXERCISES

1. Construct a triangle with side $AB = 4$ in., side $BC = 5$ in., and $\angle B$ equal to the angle of the adjacent figure.

2. Draw an acute angle A, and construct an angle twice as large as $\angle A$.

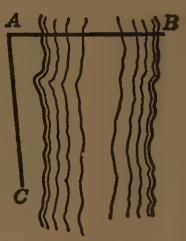
3. Construct a triangle in which the two angles adjacent to the given base are equal. Is this problem possible if the two given angles are right angles? if each is greater than a right angle?

4. The distance AB across a river can be determined without crossing it as follows: Lay off AC and locate R, the mid-point of AC. Then lay off $\angle C = \angle A$, and locate point K so that it is in a straight line with B and R. Which distance must be measured to obtain the distance across the river? Prove.



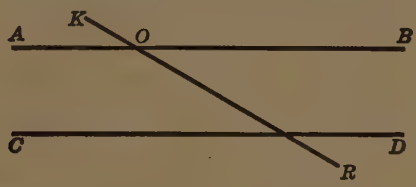
NAPOLEON AND THE CAPTAIN OF ENGINEERS

Napoleon on one of his scouting trips came to a river and asked a captain of engineers to tell him its width. The captain replied that he could guess at the width but to determine it very accurately without crossing the river and making direct measurements required the use of some surveying instruments which he did not have with him. The Emperor, so the story runs, sharply ordered the captain to determine the distance in some way at once. The latter observed that the bank where he stood sloped gently for a considerable distance parallel to the course of the river. He then stood erect at A and pulled his cap low over his eyes until point B, just at the water's edge on the opposite bank, was visible under the visor of his cap. Then, keeping rigidly the same position of head and body, he turned to face nearly at right angles to his first position. He noted carefully the point C at the water's edge just visible, as before, under his visor and stepped off the distance AC. This he gave to the Emperor as the approximate width of the river, and received immediate promotion.



Theorem 8

86. *If a line intersects one of two parallel lines, it intersects the other also.*



Given the ∥ lines *AB* and *CD* and the line *KR* cutting *AB* at *O*.

To prove that KR cuts CD.

Method. Draw through any point *O* of one line a third line. This line must intersect the second line also or there would be two lines through point *O* parallel to the second line.

Proof

1. *KR* intersects *CD* or it does not intersect it.	1. No other possibility exists.
2. If it does not, then *KR* and *CD* are ∥.	2. § 81.
3. Then *AB* and *KR* both pass through *O* and are both ∥ to *CD*.	3. By hypothesis and step 2.
4. But this contradicts the postulate of parallels.	4. § 82.
5. Then *KR* is not ∥ to *CD*, and must intersect it.	5. Remaining alternative from step 1.

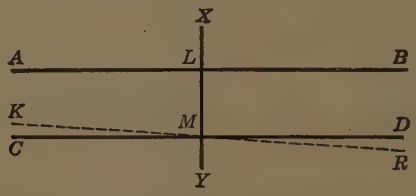
QUERY 1. What theorems from 1 to 8 are proved by superposition?

QUERY 2. On what theorems from 1 to 5 does the proof of Theorem 6 directly depend? indirectly depend?

QUERY 3. On which of the five preceding theorems does the proof of Theorem 8 directly depend? indirectly depend?

Theorem 9

87. *If a line is perpendicular to one of two parallel lines, it is perpendicular to the other also.*



Given the two ∥ lines AB and CD, with $XY \perp$ to AB at L.

To prove that XY is $\perp$ to CD.

Method. Indirect. Draw through M a line perpendicular to XY. It will be parallel to AB by a previous theorem and must coincide with CD by § 82.

Proof

1. XY intersects CD at some point, as M.	1. § 86. If a line intersects one of two ∥ lines, it intersects the other also.
2. CD is $\perp$ to XY at M or it is not $\perp$ to XY.	2. No other possibility exists.
3. Suppose CD is not $\perp$ to XY, and let KR be drawn through M $\perp$ to XY.	3. § 51, Postulate 6. At a given point on a line there is one and only one $\perp$ to it.
4. Then KR is ∥ to AB.	4. § 83. Two lines $\perp$ to the same line are ∥.
5. But this contradicts the postulate of parallels.	5. § 82. Through a given point outside a line there can be one ∥ to it and only one.
6. But KR is $\perp$ to XY.	6. By construction.
7. $\therefore$ CD is $\perp$ to XY.	7. CD coincides with KR.

Theorem 10

88. *Two right triangles are congruent if the hypotenuse and an adjacent angle of one are equal respectively to the hypotenuse and an adjacent angle of the other.*

Given rt. $\triangle ABC$ and FGH, in which the hypotenuse $AC =$ the hypotenuse FH and $\angle A = \angle F$.

To prove that $\triangle ABC \cong \triangle FGH$.

Method. Indirect. Place $\triangle ABC$ on $\triangle FGH$ so that $\angle A$ falls on $\angle F$ and point C on point H. Then CB must coincide with HG or there would be two perpendiculars from H to FG.

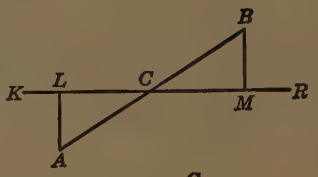
Proof

1. Place $\triangle ABC$ upon $\triangle FGH$ so that the hypotenuse AC coincides with its equal FH, and so that point A falls on F and point C on H.	1. § 51, Postulate 7.
2. Now $\angle A = \angle F$.	2. By hypothesis.
3. Hence AB will fall along FG.	3. Steps 1 and 2.
4. Then CB and HG are each $\perp$ to FG from the same point.	4. By hypothesis, $\measuredangle B$ and G are both rt. $\measuredangle$.
5. CB coincides with HG.	5. § 78. There is only one $\perp$ from a point to a line.
6. $\therefore \triangle ABC \cong \triangle FGH$.	6. The $\triangle$ coincide.

EXERCISES ON THEOREM 10

1. In the figure, C is the mid-point of AB, and AL and BM are each perpendicular to KR.

Prove $\angle A = \angle B$.

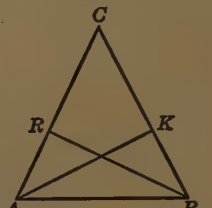


2. In the adjacent figure, K is the mid-point of AB, the base of isosceles triangle ABC. Lines KL and KM are drawn perpendicular to AC and BC.

Prove $KL = KM$.

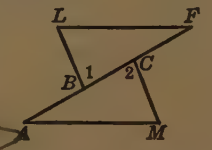
3. Prove that perpendiculars drawn from the extremities of the base of an isosceles triangle to the opposite sides are equal.

HINT. Prove either $\triangle ABK \cong \triangle ABR$ or $\triangle CAK \cong \triangle CBR$.



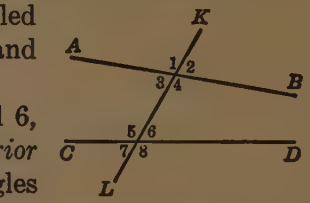
4. In the adjacent figure, $LF = AM$, $\measuredangle 1$ and 2 are right angles and $\angle F = \angle A$.

Prove $AB = CF$.



89. Transversal. A *transversal* is a line that crosses (cuts or intersects) two or more lines.

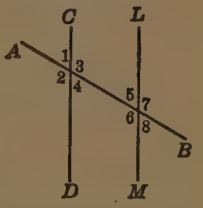
90. Angles made by a transversal. In the adjacent figure the angles 1, 2, 7, and 8 are called *exterior* angles, and angles 3, 4, 5, and 6 are called *interior* angles.



The two pairs of angles 3 and 6, 4 and 5, are called *alternate interior* angles; and the two pairs of angles 1 and 8, 2 and 7, are called *alternate exterior* angles.

The four pairs of angles 1 and 5, 2 and 6, 3 and 7, and 4 and 8 are called *corresponding* angles.

It should be noted that we now have applied the word "corresponding" to certain angles of congruent figures and to certain angles formed when a transversal cuts two other lines, which will usually be two parallel lines. Therefore, in using the word "corresponding" we should always be careful to convey the meaning we intend, by saying either "corresponding angles of parallel lines" or "corresponding angles of congruent figures."



QUERY 1. In the adjacent figure which are
(*a*) the interior angles?
(*b*) the exterior angles?
(*c*) the alternate interior angles?
(*d*) the corresponding angles?
(*e*) the alternate exterior angles?

QUERY 2. Are all four interior angles equal to each other?

QUERY 3. To what interior angle is the exterior angle 8 equal?

QUERY 4. Which angles in the figure are you now certain are equal to each other?

QUERY 5. If the transversal is perpendicular to the two parallel lines, which angles are equal to each other?

INTRODUCTION TO THEOREM 11

EXERCISES

1. Draw carefully two parallel lines and a third line cutting the two. Then measure with the protractor (*a*) the alternate interior angles, (*b*) the corresponding angles, (*c*) the alternate exterior angles.

2. Repeat Exercise 1 with a different figure.

3. Repeat Exercise 1 with still another figure.

4. Repeat Exercise 3 with two nonparallels and a transversal.

5. Complete the following statements so that they will agree with the results of Exercises 1 to 3 inclusive:

If two parallels are cut by a transversal,

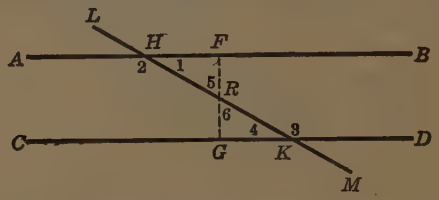
(*a*) the alternate interior angles are ____;

(*b*) the corresponding angles are ____.

(*c*) the alternate exterior angles are ____.

Theorem 11

91. *If two parallel lines are cut by a transversal, the alternate interior angles are equal.*



Given the two ∥ lines *AB* and *CD* cut by the transversal *LM* at *H* and *K* respectively, forming the pairs of alternate interior angles 1 and 4, 2 and 3.

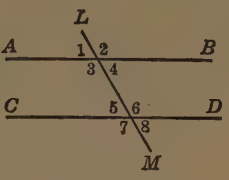
***To prove that* $\angle 1 = \angle 4$ *and* $\angle 2 = \angle 3$.**

Method. Draw *FG* through *R*, the mid-point of *KH*, perpendicular to *AB* and *CD*, and prove that $\triangle RGK$ and $\triangle RFH$ are congruent right triangles. Then $\angle 1 = \angle 4$ and $\angle 2 = \angle 3$.

Proof

1. Let *R* be the mid-point of *HK*, and let the line through *R* which is ⊥ to *AB* meet *AB* in *F* and *CD* in *G*. Then *FG* is ⊥ to *CD*.	1. § 87. If a line is ⊥ to one of two ∥ lines, it is ⊥ to the other also.
2. $RH = RK$.	2. By construction.
3. $\angle 5 = \angle 6$.	3. § 48. If two straight lines intersect, the vertical ∡ are equal.
4. $\triangle RGK \cong \triangle RFH$.	4. § 88.
5. $\angle 1 = \angle 4$.	5. § 64.
6. But $\angle 1 + \angle 2 = \angle 3 + \angle 4$.	6. § 51, Postulate 8.
7. $\angle 2 = \angle 3$.	7. § 50, Axiom 4. If equals are subtracted from equals, the results are equal.

92. ***Corollary 1.*** *If two parallel lines are cut by a transversal, the corresponding angles are equal.*



HINT. $\angle 1 = \angle 4$ and $\angle 4 = \angle 5$.

$\therefore \angle 1 = \angle 5$. Etc.

93. ***Corollary 2.*** *If two parallel lines are cut by a transversal, the sum of the two interior angles on the same side of the transversal equals two right angles.*

HINT. $\angle 3 + \angle 4 = 2$ rt. $\measuredangle$. (1)

$\angle 4 = \angle 5$. (2)

From (1) and (2), $\angle 3 + \angle 5 = 2$ rt. $\measuredangle$.

Similarly, $\angle 4 + \angle 6 = 2$ rt. $\measuredangle$.

QUERY. If the transversal is not perpendicular to one of the parallels, how many sizes of angles are formed? how many of each size?

EXERCISES

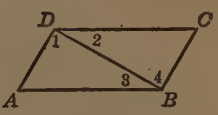
1. Prove (using the figure of § 92) that $\angle 2 = \angle 6$, $\angle 3 = \angle 7$, and $\angle 4 = \angle 8$.

2. If two parallels are cut by a transversal, the alternate exterior angles are equal (that is, in the figure above, $\angle 1 = \angle 8$, etc.).

3. State a corollary like that of § 92 concerning alternate exterior angles.

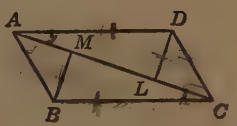
4. Prove (using the figure of § 92) $\angle 1 + \angle 7 = 2$ rt. $\measuredangle$ and $\angle 2 + \angle 8 = 2$ rt. $\measuredangle$.

5. The opposite sides of the figure $ABCD$ are parallel.



Prove $\triangle ABD \cong \triangle BCD$.

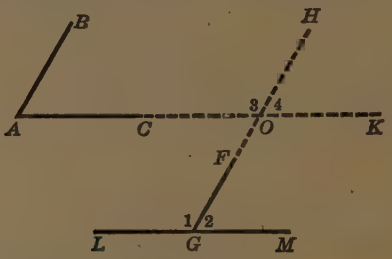
6. The opposite sides of $ABCD$ are parallel and DL and BM are each perpendicular to AC.



Prove $DL = BM$.

Theorem 12

94. *If two angles have their sides parallel each to each, they are either equal or supplementary.*



Given $AB \parallel$ to GF and $AC \parallel$ to LM.

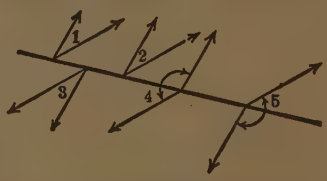
To prove that $\angle A = \angle 2$ ***and*** $\angle A$ ***is the supplement of*** $\angle 1$.

Method. Extend one side of one angle until it intersects the side of the other (extended if necessary) to which it is not parallel. Then show that one angle thus formed is equal to $\angle A$ and also equal to $\angle 2$. Then prove $\angle A$ supplementary to $\angle 1$.

Proof

1. Extend AC and GF to intersect at O.	1. § 51, Postulate 3.
2. $\angle A = \angle 4$, and $\angle 4 = \angle 2$.	2. § 92.
3. $\therefore \angle A = \angle 2$.	3. § 50, Axiom 1.
4. $\angle 2 + \angle 1 = 180°$.	4. §§ 31 and 39.
5. $\therefore \angle A + \angle 1 = 180°$.	5. § 50, Axiom 7.

The angles are equal if their sides extend in the same direction from the vertices or in the opposite direction. They are supplementary if one pair extends in the same direction and the other pair in opposite directions. The sides extend in the *same* direction if they lie on the *same* side of a straight line passing through vertices of the angles and in *opposite* directions if they lie on *opposite* sides of it.



95. Converse. If, in any theorem, the hypothesis and the conclusion be interchanged, the resulting proposition is called the *converse* of the first.

For example, the converse of Theorem 3 is: If two angles of a triangle are equal, the sides opposite them are equal.

It is important to note that the converse of a theorem is not always true.

Thus: If two triangles are congruent they have equal bases and altitudes. Converse: If two triangles have equal bases and altitudes, they are congruent. From the first two figures on the right the first theorem seems to be true, while the last two show that the converse is false. Even outside of geometry the converse of a theorem is not always true. This will be apparent from stating the converse of the following: If a man lives in Omaha, he lives in Nebraska.

EXERCISES

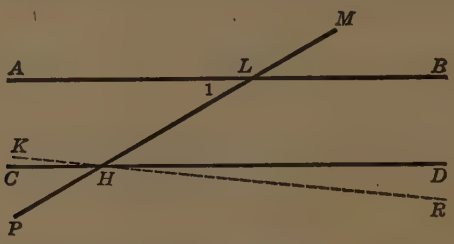
1. State the hypotheses of Theorems 2 and 3.
2. State the conclusions of Theorems 2 and 3.
3. State the converses of Theorems 2 and 12.
4. Are the converses stated in Exercise 3 true?
5. State the converses of the following statements:

(*a*) If it rains, it will turn cooler.
(*b*) If a man is rich, he is happy.
(*c*) If I eat too much, I am uncomfortable.
(*d*) If I fall in the lake, my feet become wet.
(*e*) If we walk in opposite directions, we become farther apart.
(*f*) If the sun shines, it is light.
(*g*) If my eyesight is good, I see clearly.
(*h*) If a straight-line figure has three sides, it is a triangle.

6. Are the converses (*a*) to (*h*) of Exercise 5 true?

Theorem 13 (*Converse of Theorem 11*)

96. *If two straight lines are cut by a transversal, making two alternate interior angles equal, the lines are parallel.*



Given *AB* **and** *CD* **cut by the transversal** *MP* **at** *L* **and** *H* **respectively, making alternate interior angles 1 and** *LHD* **equal.**

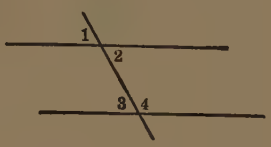
To prove that *AB* ***is*** ∥ ***to*** *CD*.

Method. Draw through *H* a line parallel to *AB*, and prove that *CD* coincides with it.

Proof

1. *CD* is ∥ to *AB* or it is not ∥ to *AB*.	1. No other possibility exists.
2. Suppose *CD* is not ∥ to *AB*. Draw *KR* through *H* ∥ to *AB*.	2. § 82.
3. Then $\angle 1 = \angle LHR$.	3. § 91. If two ∥ lines are cut by a transversal, the alternate interior ∡ are equal.
4. $\angle 1 = \angle LHD$.	4. By hypothesis.
5. $\angle LHR = \angle LHD$.	5. § 50, Axiom 1.
6. Hence *KHR* coincides with *CHD*.	6. Step 5.
7. But *KHR*, or *KR*, is ∥ to *AB*.	7. By step 2.
8. ∴ *CHD*, or *CD*, is ∥ to *AB*.	8. § 82.

97. ***Corollary 1.*** *If two lines are cut by a transversal making a pair of corresponding angles equal, the lines are parallel.*



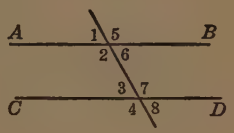
HINT. By hypothesis, $\angle 1 = \angle 3$.
$\angle 1 = \angle 2$. $\therefore \angle 2 = \angle 3$. Etc.

98. ***Corollary 2.*** *If two lines are cut by a transversal, making a pair of interior angles on the same side of the transversal supplementary, the lines are parallel.*

HINT. By hypothesis, $\angle 2 + \angle 4 = 180°$. $\angle 3 + \angle 4 = 180°$.
$\therefore \angle 2 = \angle 3$. Etc.

EXERCISES

1. In the adjacent figure, $\angle 1 = \angle 8$.
Prove AB is $\parallel$ to CD.



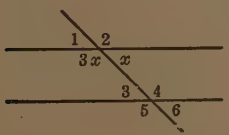
2. In the same figure, $\angle 5 = \angle 7$.
Prove AB is $\parallel$ to CD.

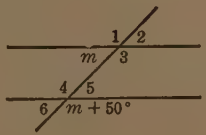
3. Also, $\angle 2 + \angle 3 = 2$ rt. $\measuredangle$.
Prove AB is $\parallel$ to CD.

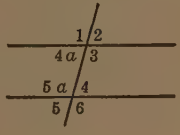
4. If in the figure for Exercise 1, $\angle 1 = 40°$, how many degrees are there in each of the other angles?

5. If in the figure for Exercise 1, $\angle 2 = (90 + x)°$, how many degrees are there in each of the other angles?

6. In each of the following figures two parallel lines are cut by a transversal. Find the number of degrees in each angle.

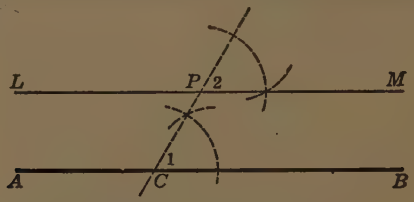






Construction 4

99. *Construct a parallel to a line through a point not on the line.*



Given line AB, and the point P not on AB.

Required to construct through P a line $\parallel$ to AB.

Method. Through P and any point C on AB draw line PC. At P construct line LPM, making $\angle 2 = \angle 1$. Then prove that LM is $\parallel$ to AB.

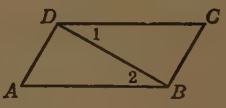
Proof

1. $\angle 1 = \angle 2$.	1. By construction.
2. $\therefore LM$ is $\parallel$ to AB.	2. § 97.

EXERCISES

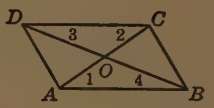
1. The opposite sides of the quadrilateral $ABCD$ are equal.

Prove DC is $\parallel$ to AB.



2. In the adjacent figure, AB is $\parallel$ to DC and equal to it.

Prove $AO = OC$ and $BO = OD$.



3. In the figure, $AO = CO$ and $BO = DO$.

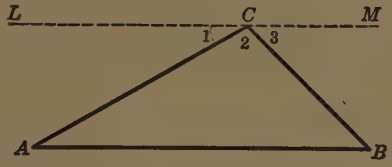
Prove $\angle 1 = \angle 2$ and AB is $\parallel$ to DC.

4. In the isosceles triangle ABC line LM is $\parallel$ to the base AB.

Prove $\angle 1 = \angle 2$.

Theorem 14

100. *The sum of the angles of any triangle is two right angles.*



Given $\triangle ABC$.

To prove that $\angle A + \angle B + \angle C = 2$ *rt.* $\measuredangle$.

Method. Draw through any vertex C a line LM parallel to the opposite side of the triangle. Then show that the sum of the three angles on one side of LM at C equals the sum of the three angles of the triangle.

Proof

1. Let LM be a line through $C \parallel$ to AB, forming $\angle 1$ and $\angle 3$ respectively.	1. § 82.
2. Then $\angle 1 + \angle 2 + \angle 3 =$ st. $\angle LCM = 2$ rt. $\measuredangle$.	2. § 50, Axiom 2; § 39.
3. $\angle 1 = \angle A$.	3. § 91. If two $\parallel$ lines are cut by a transversal, the alternate interior $\measuredangle$ are equal.
4. Also, $\angle 3 = \angle B$.	4. § 91.
5. $\angle A + \angle 2 + \angle B = 2$ rt. $\measuredangle$, or $\angle A + \angle B + \angle C = 2$ rt. $\measuredangle$.	5. Substituting $\angle A$ for $\angle 1$ and $\angle B$ for $\angle 3$ in step 2; § 50, Axiom 7.

It should be observed that the axioms used from time to time apply to numbers. If, then, for example, we say subtract line AB from line CD or substitute $\angle x$ for $\angle a$, it is the numeric measure of the line or the angle respectively to which we refer and to which the axiom applies.

QUERY 1. Can a triangle have two obtuse angles? two right angles?

QUERY 2. What relation exists between the two acute angles of a right triangle?

QUERY 3. If, in $\triangle ABC$, $\angle A = 50°$ and $\angle B = 60°$, how many degrees are there in $\angle C$?

QUERY 4. If, in $\triangle ABC$, $\angle A = 42°$ and $\angle B = \angle C$, how many degrees are there in each angle?

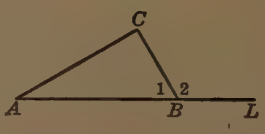
QUERY 5. If a triangle is equiangular, how many degrees are there in each of its angles?

101. Exterior angle. An *exterior angle* of a triangle is the angle between any side and the adjacent side produced. See $\angle 2$ of § 102.

QUERY. How many exterior angles has a triangle?

102. ***Corollary 1.*** *An exterior angle of a triangle is equal to the sum of the two opposite interior angles.*

HINT. $\angle 1 + \angle 2 = 2$ rt. $\measuredangle$. Why?
$\angle 1 + \angle A + \angle C = 2$ rt. $\measuredangle$. Why?
Hence $\angle 1 + \angle 2 = \angle 1 + \angle A + \angle C$.
Why?
$\therefore \angle 2 = \angle A + \angle C$. Why?



QUERY. The sum of the six exterior angles of a triangle is how many right angles?

103. ***Corollary 2.*** *If two angles of one triangle equal respectively two angles of another, the third angle of the first equals the third angle of the second.*

HINT. Let the angles of the two triangles be denoted by a, b, c, and x, y, z, respectively, where $a = x$ and $b = y$.

Then $a + b + c = x + y + z$. Why?
Also, $a + b = x + y$. Why?
$\therefore c = z$. Why?

104. ***Corollary 3.*** *The two acute angles of a right triangle are complementary.*

105. ***Corollary 4.*** *Two right triangles are congruent if a side adjacent to the right angle and an acute angle of the first are equal respectively to a side adjacent to the right angle and a corresponding angle of the second.*

HINT. By § 103, the second acute angle of the first is equal to the second acute angle of the second. ∴ a. s. a. = a. s. a., etc.

NUMERIC EXERCISES ON ANGLES OF TRIANGLES

1. If two angles of a triangle are 46° and 68° respectively, find the other angle.

2. One acute angle of a right triangle is 34° 28′. Find the other acute angle.

3. One angle of a triangle is 43° 28′ 12″; another is 81° 24′ 53″. Find the third angle.

4. How many degrees are there in each acute angle of a right isosceles triangle?

5. How many degrees are there in each exterior angle of an equiangular triangle?

6. One angle of a triangle is 64°; another is 37°. Find the six exterior angles of the triangle.

7. The vertex angle of an isosceles triangle is 56°. Find the other angles.

8. One base angle of an isosceles triangle is 24°. Find the other angles.

9. One base angle of an isosceles triangle is one third of the vertex angle. Find the number of degrees in each angle of the triangle.

10. One angle of a triangle is twice another, and the third is twice the sum of the first two. Find each angle.

11. The vertex angle of an isosceles triangle is twice the sum of the base angles. Find all three angles.

12. One exterior angle at the base of an isosceles triangle contains 110°. Find the three angles of the triangle.

Solve for x and y, and check:

13. $x + 2y = 18$,
$3x - 2y = 22$.

14. $5x + 3y = 9$,
$4x - 6y = 52$.

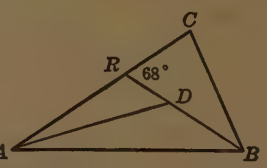
15. Solve for $\angle A$, $\angle B$, and $\angle C$:

$$\angle A + \angle B + \angle C = 180°,$$
$$2\angle A - \angle B = 76°,$$
$$\angle A - \angle B + \angle C = 84°.$$

16. In $\triangle ABC$, $\angle A + \angle B = 115°$; $\angle A + \angle C = 120°$. Find $\measuredangle A$, B, and C.

17. The difference of two angles of a triangle is 24°. The sum of one of these and the third angle is 112°. Find each angle of the triangle.

18. In $\triangle ABC$, $\angle B = 2\angle A$. The bisectors of these angles meet at D, and BD produced meets AC at R. $\angle BRC = 68°$. Find the angles of $\triangle ABC$.

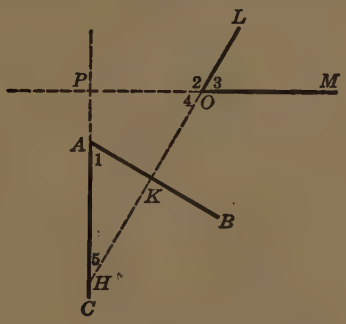


19. The side AB of any $\triangle ABC$ is extended to K. The bisectors of $\angle CBK$ and $\angle A$ meet at R. Assign a numeric value to each angle of $\triangle ABC$. Then determine the number of degrees in $\angle R$. Compare this with $\angle C$.

20. Assign other values to the angles of $\triangle ABC$ and repeat the work of the preceding exercise. Is a general conclusion indicated? If so, state it.

Theorem 15

106. *If two angles have their sides perpendicular each to each, they are equal or supplementary.*



Given $\angle CAB$ **and** $\angle MOL$, **with** $CA \perp$ **to** MO **and** $BA \perp$ **to** LO.

To prove that $\angle 1 = \angle 3$, ***and*** $\angle 1 + \angle 2 = 180°$.

Method. Extend the sides of the angles so as to form two right triangles. Show that $\angle 3 = \angle 4$ and that $\angle 1$ and $\angle 4$ are both complementary to $\angle 5$. Then $\angle 1 = \angle 3$, and $\angle 2$ will be the supplement of $\angle 1$.

Proof

1. Extend CA and MO to intersect at P, and let LO intersect AB at K and AC at H.	1. § 51, Postulate 3.
2. $\angle 3 = \angle 4$.	2. § 48. If two straight lines intersect, the vertical $\measuredangle$ are equal.
3. In $\triangle HOP$, $\angle 4 + \angle 5 = 90°$.	3. § 104.
4. In $\triangle AKH$, $\angle 1 + \angle 5 = 90°$.	4. § 104.
5. $\therefore \angle 1 + \angle 5 = \angle 4 + \angle 5$.	5. § 50, Axiom 1.

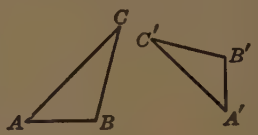
6. $\therefore \angle 1 = \angle 4 = \angle 3$.	6. § 50, Axiom 1.
7. $\angle 3 + \angle 2 = 180°$.	7. § 39.
8. $\therefore \angle 1 + \angle 2 = 180°$.	8. § 50, Axiom 7.

If in Theorem 15 both angles are acute or both are obtuse, they are equal, while if one is acute and the other obtuse they are supplementary.

EXERCISE

In the adjacent figure, AB is $\perp$ to $A'B'$, BC is $\perp$ to $B'C'$, and AC is $\perp$ to $A'C'$.

Prove $\angle A = \angle A'$, $\angle B = \angle B'$, and $\angle C = \angle C'$. Are the two triangles necessarily congruent?

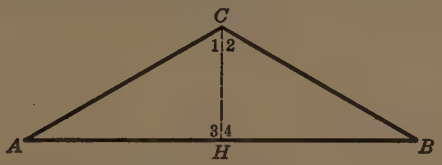


Study of the Demonstrated Theorems

After some progress has been made in geometry the student is urged to try to work out proofs of even the demonstrated theorems for himself. This can be done by covering up the proofs of these theorems and devising his own proof, guided by the "method" given in each. Later he can cover the method also and, fixing his attention on the hypothesis, the figure, and the conclusion, build up a complete proof. Many demonstrations in the text are easy, and the student will be surprised at the number he can work out independently as described. Moreover, he may often find his own proof correct but different from that of the text. He must not be discouraged if occasionally, or even frequently, he fails and must fall back on the proof given in the text. The attitude of mind which regards the demonstrated theorems as original theorems to be proved is a fine one. Persistence in the method of studying here described will give an increasing amount of mental satisfaction and insure rapid progress in geometry.

Theorem 16 (*Converse of Theorem 3*)

107. *If two angles of a triangle are equal, the sides opposite those angles are equal.*



Given $\triangle ABC$, in which $\angle A = \angle B$.

To prove that $AC = BC$.

Method. Draw CH perpendicular to AB, and then show that $\triangle AHC \cong \triangle BHC$ by a.s.a. = a.s.a. Then AC and BC are corresponding parts of congruent triangles.

Proof

1. Draw $CH \perp$ to AB.	1. § 76.
2. In $\triangle AHC$ and BHC, $CH = CH$.	2. Identical.
3. $\angle 3 = \angle 4 = 90°$.	3. By construction.
4. $\angle A = \angle B$.	4. By hypothesis.
5. $\therefore \angle 1 = \angle 2$.	5. § 103.
6. $\triangle AHC \cong \triangle BHC$.	6. § 66. a. s. a. = a. s. a.
7. $\therefore AC = BC$.	7. § 64.

108. *Corollary. If a triangle is equiangular, it is also equilateral.*

QUERY. Why was Theorem 16 not stated and proved immediately after Theorem 3?

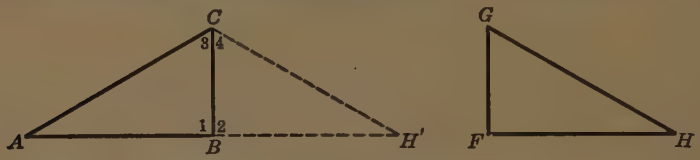
EXERCISES

1. Prove that in an isosceles triangle a line cutting any two sides and parallel to the third side forms another isosceles triangle.

2. Prove that the exterior angles at the base of an isosceles triangle are equal.

Theorem 17

109. *Two right triangles are congruent if the hypotenuse and another side of the first are equal respectively to the hypotenuse and another side of the second.*



Given rt. $\triangle ABC$ and FGH, in which the hypotenuse $AC =$ the hypotenuse HG, and $BC = FG$.

To prove that $\triangle ABC \cong \triangle FGH$.

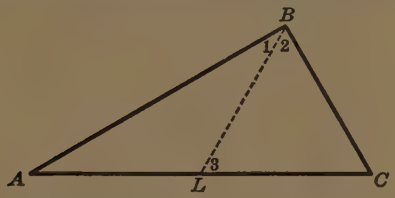
Method. Place the two triangles adjacent so that the two equal sides adjacent to the right angles coincide, making the entire figure thus formed an isosceles triangle. Then prove that the two triangles composing it are congruent.

Proof

1. Place $\triangle FGH$ adjacent to $\triangle ABC$ so that FG coincides with its equal, BC, point G falling on C and point F on B. Point H will then fall at some point H'.	1. § 51, Postulate 7.
2. $\angle 1 = \angle 2 =$ a rt. $\angle$.	2. By hypothesis.
3. $\angle 1 + \angle 2 =$ a st. $\angle$.	3. § 39.
4. $\therefore ABH'$ is a straight line.	4. Step 3.
5. $ABH'C$ is a $\triangle$.	5. § 53.
6. In $\triangle AH'C$, $AC = H'C$.	6. By hypothesis.
7. $\therefore \angle A = \angle H'$.	7. § 67.
8. Hence $\triangle ABC \cong \triangle H'BC$.	8. § 88.
9. $\therefore \triangle ABC \cong \triangle FGH$.	9. Steps 1 and 8.

Theorem 18

110. *If one acute angle of a right triangle is thirty degrees, the side opposite is half the hypotenuse.*



Given rt. $\triangle ABC$, in which $\angle A = 30°$ and AC is the hypotenuse.

To prove that $BC = \frac{1}{2}\ AC$.

Method. Draw BL, making $\angle LBA = 30°$, and prove that $AL = BL$. Then prove $\triangle BLC$ equiangular and hence equilateral. It will follow that $BL = AL = LC = BC$ and that $BC = \frac{1}{2}\ AC$.

Proof

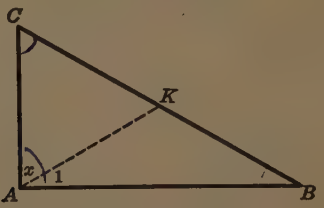
1. $\angle A = 30°$.	1. By hypothesis.
2. $\therefore \angle C = 60°$.	2. § 104.
3. Draw BL, making $\angle 1 = \angle A = 30°$.	3. § 85.
4. $AL = BL$.	4. § 107.
5. $\angle 2 = 60°$.	5. By step 3, $\angle 1 = 30°$, and by hypothesis, $\angle 1 + \angle 2 = 90°$.
6. $\angle 3 = 60°$.	6. § 102.
7. $BL = LC = BC$.	7. § 108.
8. $AL = LC = \frac{1}{2}\ AC$.	8. Steps 4 and 7 and Axiom 1.
9. $\therefore BC = \frac{1}{2}\ AC$.	9. § 50, Axiom 1.

EXERCISE

From the mid-point M of side AB of the equilateral triangle ABC a line MP is drawn perpendicular to BC at P. Prove that MP is half the altitude of the triangle.

Theorem 19

111. *If one side of a right triangle is half the hypotenuse, the angle opposite that side is thirty degrees.*



Given rt. $\triangle ABC$, in which $AC = \frac{1}{2} BC$.

To prove that $\angle B = 30°$.

Method. Draw AK, making $\angle CAK = \angle C$. Then prove that $AK = BK$ and $AK = CK$ and that $\triangle AKC$ is equilateral. The theorem then follows.

Proof

1. Draw AK, making $\angle x = \angle C$.	1. § 85.
2. Then $\angle 1 = 90° - \angle x$.	2. By hypothesis and step 1.
3. $\angle B = 90° - C = 90° - x$.	3. § 104.
4. $\therefore \angle 1 = \angle B$.	4. § 50, Axiom 1, and steps 2 and 3.
5. $AK = BK$.	5. § 107.
6. Also, $AK = CK$.	6. Step 1 and § 107.
7. $AK = \frac{1}{2} BC$.	7. Steps 5 and 6.
8. $AK = AC = CK$.	8. By hypothesis and steps 5, 6, and 7.
9. $\therefore \triangle AKC$ is equiangular.	9. § 70.
10. $\therefore \angle C = 60°$, and $\angle B = 30°$.	10. §§ 100, 104.

112. Methods of proof. Three methods of proof essentially different have been used in the theorems thus far.

1. *Proof by superposition.* This method proves that two figures are congruent by placing one upon the other and showing that the two figures must coincide. Theorem 1 was proved by superposition and Theorem 4 by a method closely akin to it.

2. *Direct demonstration.* This method consists of proving a series of dependent statements, the last of which includes the conclusion. Theorems 3, 11, and 12 were demonstrated by the direct method.

3. *Indirect demonstration.* The essential feature of the indirect method consists in assuming that the theorem to be demonstrated is not true and then showing that this assumption leads to an impossible result. The method of proving Theorems 5 and 6 is indirect. Other methods of indirect proof will be given later.

QUERY 1. In what theorems has the method of superposition been used thus far?

QUERY 2. What theorems other than 3, 11, and 12 have been demonstrated directly?

QUERY 3. What theorems thus far have been demonstrated indirectly?

QUERY 4. What is the converse of Theorem 3? Can it be demonstrated by means of the first two theorems?

113. Original demonstrations. The development of the ability to devise demonstrations of unproved theorems is one of the chief aims of the study of geometry. Much, though not all, of the work of Book I consists in proving two lines equal or two angles equal. Sometimes only a single proposition is needed to do this. Demonstrations requiring the use of congruent triangles are naturally longer and more difficult. The following suggestions should be followed in attacking original demonstrations:

1. Master the content of the previous theorems, corollaries, definitions, and axioms. This is essential.

2. Read the theorem carefully, making sure that each word is perfectly understood. Write a statement of what is given and what is to be proved.

3. Draw as accurately as possible a large figure which the statement of the theorem shows to be necessary.

Draw a general figure. If the theorem says "triangle," avoid drawing a right, an isosceles, or an equilateral triangle. If the theorem says "quadrilateral," do not draw a parallelogram or a trapezoid. Lines equal by hypothesis should really appear to be equal, parallel lines parallel, perpendicular lines perpendicular, equal angles equal. The bisector of an angle should look as if it actually bisected the angle.

Accuracy is especially desirable because a well-drawn figure often shows relations among its various lines and angles which are of great service in suggesting a proof. These relations are likely to be obscured by a poorly drawn figure.

4. Select two triangles which appear to be congruent and which contain the lines or angles to be proved equal.

5. To form the two triangles needed, it is sometimes necessary to draw lines not mentioned in the theorem, as suggested in § 72. If there is no pair of triangles which can be employed to work out a proof, try to discover a pair of auxiliary triangles which may be used as suggested in § 73.

6. Prove the triangles congruent by the use of one of the following theorems: 1, 2, 4, 10, 17, and § 105.

7. Draw the conclusion: The required lines or angles are equal because they are corresponding parts of congruent figures.

8. If the preceding steps fail, plan the solution anew with a different figure. Systematic planning and persistence are essential to success in original demonstrations.

While mastery of direct methods and the use of congruent triangles is very important, indirect methods and

superposition may at times be necessary. In fact, the key to the solution of an original may not be one of the theorems on congruent triangles at all, but may be some one or more of the other demonstrated theorems.

114. The drawing of auxiliary lines. Auxiliary lines must frequently be added to the figure to aid in the demonstration. Usually such lines may be drawn to satisfy only *two* conditions, as follows:

(*a*) Joining two points.
(*b*) Extending a given line some definite length.
(*c*) Bisecting a given angle.
(*d*) Through a point and parallel to a given line.
(*e*) Through a point and perpendicular to a given line.
(*f*) Intersecting a given line and making an angle with it equal to a given angle.

QUERY 1. Can a line be drawn through a given point parallel to one given line and at the same time perpendicular to another given line?

QUERY 2. Can a line be drawn through the vertex of any given triangle bisecting the angle at that vertex and at the same time perpendicular to the opposite side?

QUERY 3. Can a line be drawn from one vertex of any given triangle bisecting the opposite side and at the same time perpendicular to it?

QUERY 4. Can a line be drawn through the mid-point of one side of any given triangle and at the same time bisecting the angle opposite that side?

QUERY 5. Can one diagonal of any given quadrilateral be drawn so as to bisect the other?

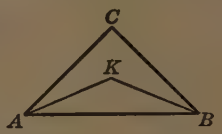
115. Extending a line. The direction in which a line segment is to be extended is determined by the order in which its end-points are named. Thus it is understood that if a line segment AB is to be prolonged beyond B, we say *extend,* or *produce, AB;* if it is to be prolonged in the other direction beyond A, we say *extend,* or *produce, BA.*

GEOMETRIC PROPOSITIONS ON RIGHT, ISOSCELES, AND EQUILATERAL TRIANGLES

1. Prove Theorem 16 by drawing CH bisecting $\angle ACB$.

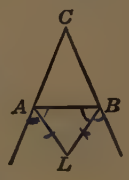
✓ 2. Prove that a line parallel to the base of an equilateral triangle forms an equilateral triangle.

3. The bisectors of the base angles A and B of the isosceles triangle ABC meet at K. Prove $AK = BK$.



4. If one angle of an isosceles triangle is 60°, prove that the triangle is equilateral.

5. The bisectors of the exterior angles at the base of the isosceles triangle ABC form with the base an isosceles triangle ABL. Prove.

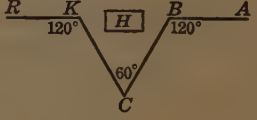


6. A line cutting two sides of an equilateral triangle so as to form another equilateral triangle is parallel to the third side. Prove.

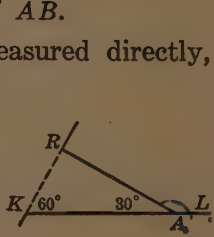
7. The vertex angle C of the isosceles triangle ABC contains 120°. A perpendicular to AB at B meets AC extended in K. Prove $\triangle BCK$ equilateral.

8. State the converse of Exercise 2 above.

9. The straight line AB may be extended through the house H as follows: Lay off $\angle ABC = 120°$ and $\angle C = 60°$. Then make $CK = BC$ and $\angle K = 120°$. Show that KR will be a continuation of AB.



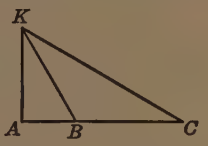
10. If the distance KR cannot be measured directly, it can be found as follows: Make $\angle RKL = 60°$. Then find a point A on KL such that $\angle RAK = 30°$. What distance must be measured to determine the length of KR? Prove.



11. The line from the vertex of an isosceles triangle to the mid-point of the base is perpendicular to the base and bisects the vertex angle. Prove.

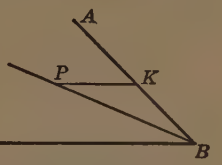
12. If a line from the vertex of an isosceles triangle is perpendicular to the base, it bisects the base and the vertex angle. Prove.

13. In the figure, $BC = 120$ ft., $\angle C = 30°$, $\angle KAB = 90°$, and $\angle ABK = 60°$. Find AB.



14. If two exterior angles of a triangle are equal, the triangle is isosceles. Prove.

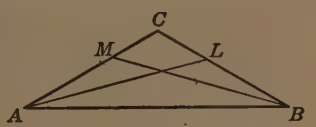
15. If a line is drawn through any point in the bisector of an angle parallel to one side of the angle and intersecting the other, prove that an isosceles triangle is formed.



16. Prove that the lines drawn from the extremities of the base of an isosceles triangle to the mid-points of the opposite sides are equal.

17. If lines drawn from the mid-point of one side of a triangle perpendicular to the other two sides are equal, the triangle is isosceles. Prove.

18. The bisectors of the base angles A and B of the isosceles triangle ABC meet the opposite sides in L and M. Prove $AL = BM$. How many pairs of triangles may be used to prove this proposition? Name them.

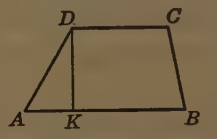


19. The bisector of the exterior angle at the vertex of an isosceles triangle is parallel to the base. Prove.

20. State and prove the converse of the preceding exercise.

21. If the base of an isosceles triangle is divided into four equal parts, prove that the lines drawn from the vertex to the two outer points of division are equal.

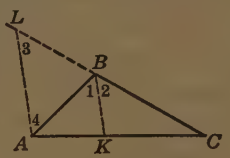
22. In figure $ABCD$, AB is $\parallel$ to CD. If DK is $\perp$ to AB at K and makes $AK = \frac{1}{2}\,AD$, find the number of degrees in $\angle ADC$.



23. The sides AB, BC, and CA of the equilateral triangle ABC are extended the same length to K, R, and L respectively. Prove that the triangle whose vertices are K, R, and L is equiangular.

24. In $\triangle ABC$ sides AB and BC are equal. AB is extended its own length to K and line KC is drawn. Prove $\angle KCA$ a right angle.

25. In $\triangle ABC$ line BK bisects $\angle ABC$. A line parallel to BK through A meets CB extended in L. Prove $AB = BL$.



116. Use of algebra in proving theorems. In the demonstration of many geometric theorems algebra can be used effectively. Unknown lines or angles of the figure may be denoted by algebraic symbols and one or more equations stated, the solution of which leads to the required conclusion.

EXAMPLE

In an isosceles triangle a line drawn from one extremity of the base perpendicular to the opposite side makes with the base an angle equal to half the vertex angle.

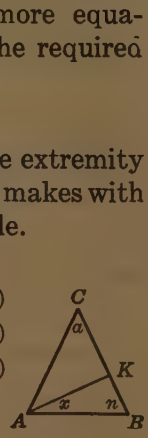
Proof. Draw $AK \perp$ to BC.

In $\triangle ABK$, $x + n = 90°$. [§ 104] (1)

In isosceles $\triangle ABC$, $a + 2n = 180°$. [§ 100] (2)

(1) $\times$ 2, $2x + 2n = 180°$. (3)

(3) $-$ (2), $2x - a = 0$ or $x = \frac{a}{2}$.

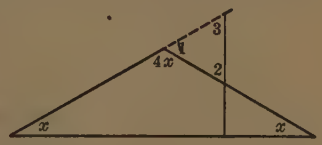


EXERCISES

1. If one acute angle of a right triangle is double the other, find each.

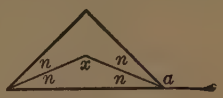
2. One angle of a triangle is twice the second and the second is three times the third. Find the three angles.

3. If each of the angles at the base of an isosceles triangle is one fourth the vertex angle, every line perpendicular to the base which does not pass through the vertex forms an equilateral triangle with the other two sides, one being produced.



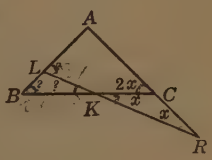
HINT. First find x. Then find $\angle 1$, $\angle 2$, and $\angle 3$.

4. The angle between the bisectors of the base angles of an isosceles triangle is equal to an exterior angle at the base of the triangle.



5. K is any point on the base BC of the isosceles triangle ABC. Side AC is produced from C to R so that $CR = CK$, and RK is produced to meet AB at L.

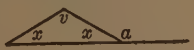
Prove $\angle ALR = 3 \angle ARL$.



6. Lines are drawn from the vertex of an isosceles triangle perpendicular to the equal sides. Prove that the angle between them equals the sum of the base angles of the triangle.

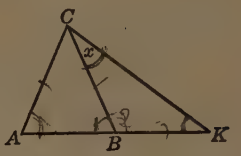
7. Prove that an exterior angle at the base of an isosceles triangle equals 90° plus half the vertex angle.

HINT. $x + a = 180°$ and $2x + v = 180°$. Eliminate x between the two equations, and a will be expressed in terms of v.



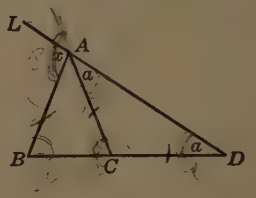
8. The base AB of an isosceles triangle is produced to K.

Prove $\angle KCB = \angle A - \angle K$.



9. In the adjacent figure, line $AB = AC = CD$.

Prove $\angle LAB \doteq 3 \angle D$.



QUADRILATERALS

117. Broken line. A *broken line* is a line composed of two or more straight line segments placed end to end.

118. Polygon. A *polygon* is a closed broken line in a plane.

119. Quadrilateral. A *quadrilateral* is a polygon which has four sides.

120. Trapezoid. A *trapezoid* is a quadrilateral with one and only one pair of sides parallel.

The *bases* of a trapezoid are the two parallel sides.

The *altitude* of a trapezoid is the perpendicular distance between the bases.

The *median* of a trapezoid is the line joining the midpoints of the nonparallel sides.

121. Parallelogram. A *parallelogram* is a quadrilateral whose opposite sides are parallel.

An altitude of a parallelogram is the perpendicular distance between either pair of opposite sides.

122. Rectangle. A *rectangle* is a parallelogram whose angles are all right angles.

123. Square. A *square* is a rectangle whose sides are equal.

124. Rhombus. A *rhombus* is a parallelogram whose sides are equal and whose angles are not right angles.

Introduction to Parallelograms

EXERCISES

1. Draw a quadrilateral in which the opposite sides are parallel and whose angles are not right angles. Then measure (*a*) the opposite sides, (*b*) the opposite angles, (*c*) the two diagonals, (*d*) the two segments of each diagonal, (*e*) the two angles formed by the diagonals at each vertex. Tabulate the results of this exercise with those of the next four.

2. Draw a different figure and repeat Exercise 1.

3. Draw a rhombus and repeat Exercise 1.

4. Draw a rectangle and repeat Exercise 1.

5. Draw a square and repeat Exercise 1.

6. Study the tabulated results and complete the following statements to agree with them:

(*a*) The opposite sides of a parallelogram are _____.

(*b*) The opposite angles of a parallelogram are _____.

(*c*) The diagonals of a rectangle are _____.

(*d*) The diagonals of a parallelogram _____ each other.

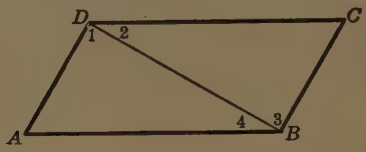
(*e*) The diagonals of every rhombus _____ its angles.

QUERY 1. Do the diagonals of a rectangle seem to have any property that the diagonals of other parallelograms do not have? What property?

QUERY 2. Do the diagonals in any parallelogram seem to be perpendicular to each other? If so, in what kind of parallelogram?

Theorem 20

125. *Either diagonal of a parallelogram divides it into two congruent triangles.*



Given $\square ABCD$.

To prove that $\triangle ABD \cong \triangle BDC$.

Method. Show that $\triangle ABD \cong \triangle BDC$ by **a.s.a. = a.s.a.**

Proof

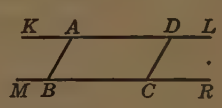
1. $\angle 1 = \angle 3$ and $\angle 2 = \angle 4$.	1. § 91.
2. $BD = BD$.	2. Identical.
3. $\therefore \triangle ABD \cong \triangle BDC$.	3. § 66. a.s.a. = a.s.a.

If diagonal AC is drawn instead of BD, a like proof holds.

126. *Corollary 1. The opposite sides of a parallelogram are equal.*

127. *Corollary 2. The opposite angles of a parallelogram are equal.*

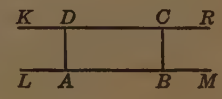
128. *Corollary 3. Parallel segments included between parallels are equal.*



HINT. What kind of figure is $ABCD$?

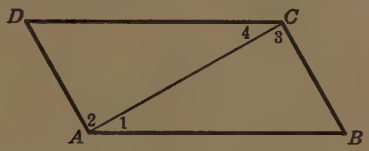
129. *Corollary 4. Two parallels are everywhere equally distant.*

HINT. AD and BC are any two perpendiculars to KR. Prove AD parallel to BC and equal to it and that each is perpendicular to LM.



Theorem 21

130. *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*



Given the quadrilateral *ABCD* with *AB* equal and parallel to *DC*.

To prove that the quadrilateral ABCD is a parallelogram.

Method. Draw AC, and prove that $\triangle ABC \cong \triangle ADC$ by s.a.s. = s.a.s. Then $\angle 2 = \angle 3$ and BC is $\parallel$ to AD.

Proof

1. Draw the diagonal AC.	1. § 51, Postulate 1.
2. AB is $\parallel$ to DC.	2. By hypothesis.
3. Hence $\angle 1 = \angle 4$.	3. § 91.
4. $AB = DC$.	4. By hypothesis.
5. $AC = AC$.	5. Identical.
6. $\triangle ABC \cong \triangle ADC$.	6. § 62. s.a.s. = s.a.s.
7. $\angle 2 = \angle 3$.	7. § 64.
8. AD is $\parallel$ to BC.	8. § 96.
9. $\therefore ABCD$ is a $\square$.	9. § 121. Definition.

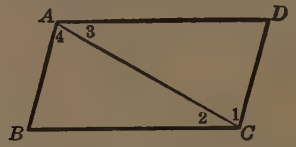
EXERCISES

1. Prove that the lines joining the mid-points of adjacent sides of a parallelogram form a parallelogram.

2. Prove that the bisectors of two adjacent angles of a parallelogram intersect each other at right angles.

Theorem 22

131. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*



Given the quadrilateral $ABCD$, in which $AB = DC$ and $AD = BC$.

To prove that ABCD is a parallelogram.

Method. Draw the diagonal AC, and prove that $\triangle ABC \cong \triangle ADC$. Then show that $\angle 1 = \angle 4$ and that $\angle 2 = \angle 3$. It follows that AB is $\parallel$ to DC and AD is $\parallel$ to BC.

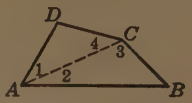
Proof

1. Draw the diagonal AC.	1. § 51, Postulate 1.
2. In the $\triangle ABC$ and ADC, $AB = DC$ and $AD = BC$.	2. By hypothesis.
3. $AC = AC$.	3. Identical.
4. $\triangle ABC \cong \triangle ADC$.	4. § 71. s.s.s. = s.s.s.
5. $\angle 1 = \angle 4$ and $\angle 3 = \angle 2$.	5. § 64.
6. DC is $\parallel$ to AB and AD is $\parallel$ to BC.	6. § 96.
7. $\therefore ABCD$ is a $\square$.	7. § 121.

EXERCISES

1. Prove that the sum of the angles of a quadrilateral is four right angles.

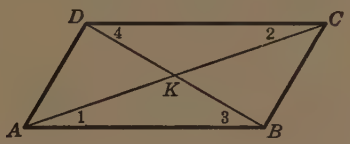
HINT. Draw AC. $\angle 1 + \angle 4 + \angle D = 2$ rt. $\triangleq$. Etc.



2. Prove that the angles adjacent to one side of a parallelogram are supplementary.

Theorem 23

132. *The diagonals of a parallelogram bisect each other.*



Given $\square ABCD$, **with its diagonals** AC **and** BD **intersecting at** K.

To prove that $AK = KC$ ***and*** $DK = KB$.

Method. Show $\triangle ABK \cong \triangle CDK$ by a.s.a. = a.s.a. Then the two segments of each diagonal are corresponding parts of congruent triangles.

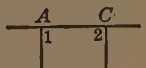
Proof

1. $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$.	1. § 91.
2. $AB = DC$.	2. § 126.
3. $\triangle ABK \cong \triangle CDK$.	3. § 66. a.s.a. = a.s.a.
4. $\therefore AK = KC$ and $DK = KB$.	4. § 64.

EXERCISES

1. Prove that if two lines are perpendicular to one of two parallels, the segments of the perpendiculars included between the two parallels are equal.

HINT. Let A and C be any two points on one of the parallels. Let AB and CD be $\perp$ to AC. $\angle 1 = ?$ $\angle 2 = ?$ Etc.

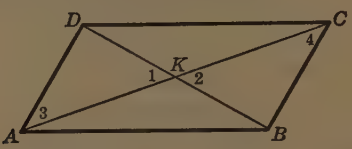


2. Prove that if both pairs of opposite angles of a quadrilateral are equal, the figure is a parallelogram.

3. Prove that if the adjacent angles of a quadrilateral are supplementary, the figure is a parallelogram.

Theorem 24 (Converse of Theorem 23)

133. *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*



Given the quadrilateral *ABCD*, in which $AK = KC$ and $DK = KB$.

To prove that the quadrilateral ABCD is a parallelogram.

Method. Prove $\triangle AKD \cong \triangle BKC$ by s.a.s. = s.a.s. Then $AD = BC$, $\angle 3 = \angle 4$, and hence AD is $\parallel$ to BC. It follows that $ABCD$ is a parallelogram.

Proof

1. $AK = KC$ and $DK = KB$.	1. By hypothesis.
2. $\angle 1 = \angle 2$.	2. § 48.
3. $\triangle AKD \cong \triangle BKC$.	3. § 62. s.a.s. = s.a.s.
4. $AD = BC$ and $\angle 3 = \angle 4$.	4. § 64.
5. AD is $\parallel$ to BC.	5. § 96.
6. $\therefore ABCD$ is a $\square$.	6. § 130.

EXERCISES

1. Prove that if one angle of a parallelogram is a right angle, the figure is a rectangle.

2. Prove that if the four angles of a quadrilateral are equal, the figure is a rectangle.

3. Prove that the diagonals of a rectangle are equal.

4. State and prove the converse of the preceding exercise.

5. Prove that the diagonals of a rhombus or a square meet at right angles and bisect the four angles of each.

REVIEW QUESTIONS

1. What common properties have a square and a rhombus?

2. How can a square be defined without using the word "rectangle"?

3. If the word "triangle" in Theorems 1 and 2 is changed to "parallelogram," will the resulting statements be true?

4. If a line is drawn parallel to the base of a triangle, what kind of figures are formed?

5. How many sides of a trapezoid may be equal to each other?

6. If three of the sides of a quadrilateral are equal, is the fourth side equal to each of the others?

7. If the diagonals of a quadrilateral are equal, what kind of quadrilateral may it be?

8. May a trapezoid have its diagonals perpendicular to each other?

9. May the diagonals of a trapezoid bisect each other?

10. What kinds of quadrilaterals have equal diagonals which bisect each other?

11. Are two equilateral triangles necessarily congruent?

12. Are two equiangular triangles necessarily congruent?

13. Are adjacent angles ever equal? When?

14. What kind of angle is equal to its supplement? less than its supplement? greater than its supplement?

15. May each of two adjacent angles be a straight angle?

16. In what kind of triangle are the three medians equal? two medians?

17. If two parallels are cut by a transversal, how many angles are formed? How many angles of different size exist among these? What names are applied to the pairs of equal angles?

18. What is the altitude of a triangle? the base?

19. How many altitudes of an obtuse triangle fall outside the triangle? How many of a right triangle? How many of an acute triangle?

20. In what kind of triangle does an altitude bisect one of the angles? each of the altitudes bisect an angle?

21. In what kind of triangle may a median, an altitude, and a bisector of an angle coincide?

22. Do the altitudes and medians of a triangle ever coincide with the bisectors of its angles?

23. Are alternate exterior angles of parallel lines always equal?

24. What is the sum of the angles adjacent to one base of a parallelogram? of a trapezoid?

25. State five facts concerning a parallelogram.

26. In what kind of parallelogram are the diagonals equal?

27. What two kinds of parallelogram have diagonals perpendicular to each other?

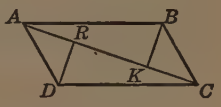
28. For what quadrilaterals have altitudes been defined? Do any quadrilaterals have more than one altitude?

29. What quadrilaterals have more than one base? How many altitudes does each quadrilateral have?

30. In what kind of quadrilateral are the altitudes equal to the sides?

GEOMETRIC PROPOSITIONS ON TRIANGLES AND PARALLELOGRAMS

1. In the parallelogram $ABCD$ lines are drawn from B and D, the ends of the shorter diagonal, perpendicular to AC and meeting it in K and R respectively.



Prove that the angles of $\triangle BKC$ are equal respectively to the angles of $\triangle DRA$, and that $BK = DR$.

2. $ABCD$ is a parallelogram. AB and CD are produced in opposite directions the same distance to K and R respectively.

Prove $\triangle ADR \cong \triangle CBK$.

3. A line joining the mid-points of two opposite sides of a parallelogram divides it into two parallelograms. Prove.

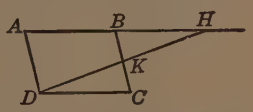
4. AC is the longer diagonal of the parallelogram $ABCD$. From C and D perpendiculars are drawn to AB and AB produced, meeting it in K and R respectively.

Prove that $\triangle BKC$ and DRA are mutually equiangular, and that $CK = DR$.

5. $ABCD$ is a parallelogram, and O is the mid-point of the diagonal AC. A line is drawn through O cutting DC in K and AB in R.

Prove $KO = OR$.

6. K is the mid-point of BC in the parallelogram $ABCD$. Line DK produced intersects AB produced in H.



Prove $\triangle KCD \cong \triangle KBH$.

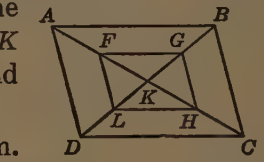
7. In the quadrilateral $ABCD$ the diagonals intersect at K so that $KA = KB$ and $KC = KD$.

Prove $BC = AD$.

8. The mid-point of one side of a parallelogram is joined to a vertex not adjacent to the side, and the mid-point of the opposite side is joined to the vertex opposite the first.

Prove that the two lines thus determined are equal and parallel.

9. AC and BD, the diagonals of a parallelogram, meet in K. Points F, G, H, and L are the mid-points of AK, BK, CK, and DK respectively. Lines FG, GH, HL, and LF are drawn.



Prove that $FGHL$ is a parallelogram.

10. If two opposite sides of a parallelogram are produced the same distance in opposite directions and their ends joined to corresponding vertices so as to form a quadrilateral, it will be a parallelogram. Prove.

11. $ABCD$ is a parallelogram. R is taken on AB and K on CD so that $BR = DK$.

Prove that $AKCR$ is a parallelogram.

12. If a triangle is isosceles, the lines from the extremities of the base bisecting the equal angles and terminating in the opposite sides are equal.

13. State the converse of the preceding theorem.

NOTE. The student will observe that he is not asked to prove the theorem just stated. The theorem is true and its proof may appear to be easy. It is really difficult to prove it, however, if one is limited to the theorems of Book I.

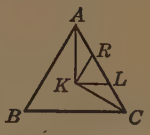
14. If the angles adjacent to the longer of the two parallel sides of a trapezoid are equal, the nonparallel sides are equal. Prove.

HINT. What two lines forming two right triangles may be of use? or what single line forming a parallelogram?

15. One angle formed by the bisectors of two angles of an equilateral triangle is double the third angle of the triangle. Prove.

HINT. It is frequently helpful, as here, to determine the number of degrees in each angle of a figure and to write it plainly within the angle. Often the proof of the theorem will then be obvious.

16. ABC is an equilateral triangle. The bisectors of $\measuredangle A$ and C meet at K. The line KR is drawn parallel to AB, meeting AC at R, and KL is drawn parallel to BC, meeting AC at L.



Prove that AR, RK, RL, KL, and CL are equal.

17. The line through the vertex of an isosceles triangle parallel to the base bisects the exterior angles at the vertex. Prove.

18. Has the theorem of Exercise 17 a converse? more than one? If so, state them.

19. Under what conditions may a theorem have more than one converse?

20. ABC is any triangle. AKB and ARC are equilateral triangles adjacent to $\triangle ABC$ and having the sides AB and AC, respectively, in common with it.

Show that $CK = BR$.

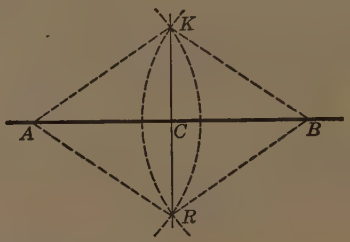
134. Mid-perpendicular. If a line is perpendicular to another line at its middle point, the first line is called the *mid-perpendicular* or the *perpendicular bisector* of the second.

QUERY 1. Is the altitude of a triangle necessarily the mid-perpendicular of the base?

QUERY 2. In what kind of triangles is the answer to Query 1 affirmative?

Construction 5

135. *Construct the perpendicular bisector of a given line segment.*



Given the line segment *AB*.

Required to construct the perpendicular bisector of AB.

Method. Draw two arcs with A and B as centers and with a radius great enough to give two points of intersection, K and R. Draw KR cutting AB at C. Then $AC = BC$, and $\angle KCA = \angle KCB = 90°$.

Proof

1. $AK = BK$ and $AR = BR$.	1. By construction.
2. $\therefore AC = BC$ and $\angle KCA = \angle KCB = 90°$.	2. § 77. If two points are equally distant from the extremities of a line segment, they determine its ⊥ bisector.

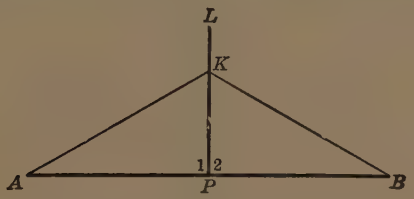
QUERY 1. How many points on KR are just as far from A as from B?
QUERY 2. If a longer radius than AK had been taken, would the resulting line have coincided with KR?
QUERY 3. Is this construction always possible?
QUERY 4. In § 76 is KM the mid-perpendicular of the line segment RL?

EXERCISES

1. Draw a line CD and divide it into four equal parts.
2. Draw a line EF and divide it into eight equal parts.

Theorem 25

136. *If a point is on the mid-perpendicular of a line, it is equidistant from the ends of the line.*



Given $LP \perp$ to AB, $PA = PB$, K any point on LP, and the lines KA and KB.

To prove that $KA = KB$.

Method. Prove $\triangle APK \cong \triangle BPK$ by s.a.s. = s.a.s.

Proof

1. In $\triangle APK$ and $\triangle BPK$, $PA = PB$.	1. By hypothesis.
2. $KP = KP$.	2. Identical.
3. $\angle 1 = \angle 2$.	3. By hypothesis and § 35.
4. $\triangle APK \cong \triangle BPK$.	4. § 62. s.a.s. = s.a.s.
5. $\therefore KA = KB$.	5. § 64.

QUERY 1. If, in the adjacent figure, KR is the mid-perpendicular of AB and LM is the mid-perpendicular of BC, what is true of their point of intersection with respect to points A, B, and C?

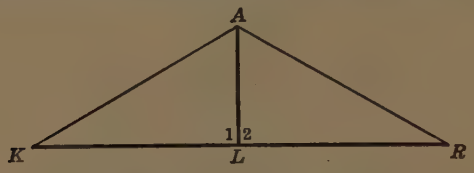
QUERY 2. Is there a point equidistant from the three vertices of any triangle? If so, how could it be found?

QUERY 3. Is there a point equidistant from all four vertices of a parallelogram?

QUERY 4. Is there any point equidistant from all four vertices of a rectangle?

Theorem 26 (*Converse of Theorem 25*)

137. *If a point is equidistant from the ends of a line, it is on the mid-perpendicular of the line.*



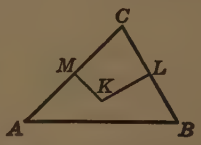
Given the line KR and the point A so that $AK = AR$.

To prove that A lies on the mid-perpendicular of KR.

Method. Draw line AL perpendicular to KR, and show that $\triangle AKL \cong \triangle ARL$ by § 109. Then $KL = RL$ and $\angle 1 = \angle 2$.

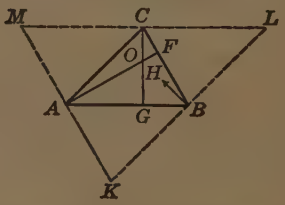
The proof is left to the student.

138. *Corollary 1. The perpendicular bisectors of the sides of a triangle intersect in a common point.*



HINT. Let the mid-perpendiculars of AC and BC intersect at K, and prove K equidistant from A and B.

139. *Corollary 2. The altitudes of a triangle intersect in a common point.*



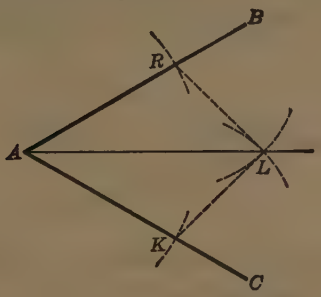
HINT. Draw through A, B, and C lines parallel to the opposite sides and intersecting in K, L, and M. Prove that A, B, and C are the mid-points of MK, KL, and LM respectively. Then use § 138.

QUERY 1. At what point do the altitudes of an isosceles right triangle meet?

QUERY 2. Where are all the points equally distant from two parallel lines?

Construction 6

140. *Bisect a given angle.*



Given $\angle BAC$.

Required to bisect $\angle BAC$.

Method. With A as the center and any radius AR, draw two arcs cutting the sides of $\angle BAC$ at R and K. Then with any radius greater than half RK and with R and K as centers, draw two arcs intersecting at L. Draw AL. Then prove $\angle BAL = \angle CAL$.

Proof

1. Draw LR and LK.	1. § 51, Postulate 1.
2. $AR = AK$ and $LR = LK$.	2. By construction.
3. $AL = AL$.	3. Identical.
4. $\triangle ALR \cong \triangle ALK$.	4. § 71. s.s.s. = s.s.s.
5. $\therefore \angle BAL = \angle CAL$.	5. § 64.

EXERCISES

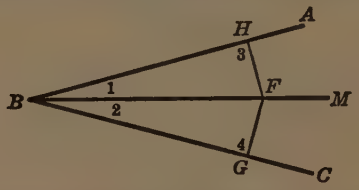
1. Divide an angle EFG into four equal parts.

2. Draw a large triangle and bisect each of its angles. Do the bisectors meet at a point?

3. Draw a pair of supplementary adjacent angles and bisect each angle. What is the angle between the bisectors? Measure it with the protractor.

Theorem 27

141. *If a point is on the bisector of an angle, it is equally distant from the sides of the angle.*



Given $\angle ABC$ **with** BM **its bisector,** F **a point on** BM**, and** FH **and** FG **the distances from** F **to** AB **and** CB **respectively.**

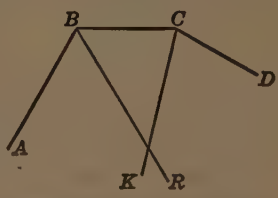
To prove that $FH = FG$.

Method. Prove that $\triangle BFH \cong \triangle BFG$. Then FH and FG are corresponding parts of congruent triangles.

Proof

1. In $\triangle BFH$ and $\triangle BFG$, $\angle 1 = \angle 2$.	1. By hypothesis.
2. $\angle 3 = \angle 4 = 90°$.	2. § 79. Definition of distance from a point to a line.
3. $BF = BF$.	3. Identical.
4. $\triangle BFH \cong \triangle BFG$.	4. § 88.
5. $\therefore FH = FG$.	5. § 64.

QUERY 1. In the adjacent figure, BR and CK bisect $\measuredangle B$ and C respectively. What is true of the position of the point where BR and CK intersect?

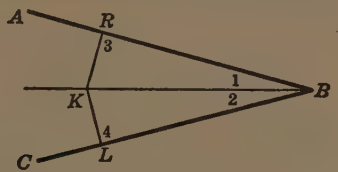


QUERY 2. In what kind of parallelogram do the diagonals bisect its angles? What is true of the position of the intersection point of the diagonals in such figures?

QUERY 3. In an equilateral triangle, what three sets of lines, three in each set, intersect in the same point?

Theorem 28 (Converse of Theorem 27)

142. *If a point is equally distant from the sides of an angle, it is on the bisector of the angle.*



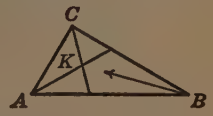
Given $\angle ABC$ **and point** K **such that** $\perp$**s** KR **and** KL **are equal.**

To prove that the point K is on the bisector of $\angle ABC$.

Method. Draw BK, and then prove by § 109 that $\triangle BKL \cong \triangle BKR$. Then $\angle 1 = \angle 2$, being corresponding parts of congruent triangles.

The proof is left to the student.

143. ***Corollary.*** *The bisectors of the three angles of a triangle intersect in a common point.*



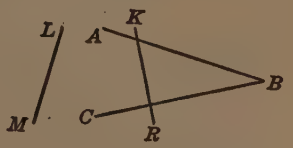
HINT. Let the bisectors of $\angle A$ and $\angle C$ intersect at K, and prove K equidistant from BA and BC.

QUERY 1. Is the point of intersection of the bisectors of two angles of a triangle equidistant from the sides of the triangle?

QUERY 2. If a point is equidistant from the three sides of a triangle, on the bisectors of what angles does it lie?

QUERY 3. Do the bisectors of the angles of a rhombus intersect in the same point?

QUERY 4. Do the bisectors of the angles of every parallelogram intersect in a common point?

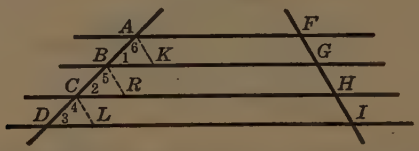


QUERY 5. Is there a point on line KR which is equidistant from AB and BC? How could the point be found? Is this true for LM?

QUERY 6. What use could you make of § 143 in inscribing a circle in a given triangle?

Theorem 29

144. *If three or more parallels intercept equal segments on one transversal, they intercept equal segments on any other transversal.*



Given the ∥ lines *AF*, *BG*, *CH*, and *DI* intercepting the equal segments *AB*, *BC*, and *CD* on the transversal *AD* and intercepting the segments *FG*, *GH*, and *HI* on another transversal.

***To prove that* $FG = GH = HI$.**

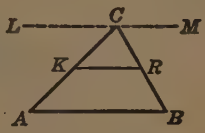
Method. Draw through *A*, *B*, and *C* lines parallel to *FI*, and prove ≙ *ABK*, *BCR*, *CDL* congruent triangles. Then $AK = BR = CL$. Next from the parallelograms formed prove that these lines equal *FG*, *GH*, and *HI*.

Proof

1. Through *A*, *B*, and *C* draw ∥s to *FI*, meeting *BG*, *CH*, and *DI* in *K*, *R*, and *L* respectively.	1. § 82.
2. Then *AK*, *BR*, and *CL* are ∥ lines.	2. § 84.
3. In ≙ *ABK*, *BCR*, and *CDL*, $\angle 6 = \angle 5 = \angle 4$.	3. § 92.
4. Also, $\angle 1 = \angle 2 = \angle 3$.	4. § 92.
5. $AB = BC = CD$.	5. By hypothesis.
6. $\triangle ABK \cong \triangle BCR \cong \triangle CDL$.	6. § 66. a.s.a. = a.s.a.

7. $AK = BR = CL$. 7. § 64.
8. But $AK = FG$, $BR = GH$, and $CL = HI$. 8. § 126.
9. $\therefore FG = GH = HI$. 9. § 50, Axiom 1.

145. *Corollary. If a line bisects one side of a triangle and is parallel to another side, it bisects the third side.*



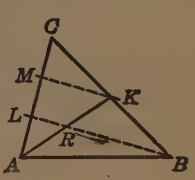
HINT. Let K be the mid-point of AC and let KR be parallel to AB. Draw LM through C parallel to AB. Then LM, KR, and AB are parallel lines. Therefore $CR = RB$.

EXERCISES

1. Given a parallelogram $ABCD$. The sides AD and BC are divided into five equal parts, and the corresponding points of division are joined. Show that these lines divide the diagonals of the parallelogram into five equal parts.

2. If a line bisects one nonparallel side of a trapezoid and is parallel to one base, it bisects the other nonparallel side also. Prove.

3. In the adjacent figure, ABC is a triangle. The point K is the mid-point of BC and R is the mid-point of AK. BR produced meets AC in L.



Prove $AL = \frac{1}{3}\ AC$.

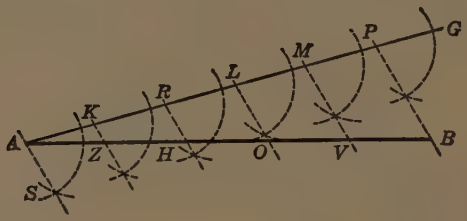
4. The point K is taken on AB of the parallelogram $ABCD$ so that AK is one fifth of AB. The lines DK and AC intersect at R.

Prove $AR = \frac{1}{6}\ AC$.

HINT. Divide DC into five equal parts, and through the points of division draw lines parallel to DK and intersecting AB.

Construction 7

146. *Divide a given straight line into three or more equal parts.*



Given the line AB.

Required to divide the line AB into (say) five equal parts.

Method. Draw AG, making $\angle BAG$ about 30°. Take AK on AG approximately equal to one fifth of AB, and lay off on AG in succession KR, RL, LM, and MP, each equal to AK. Draw BP. Then through A, K, R, L, and M respectively construct lines parallel to PB, cutting AB in A, Z, H, O, and V respectively. Then $AZ = ZH = HO = OV = VB$.

Proof

1. $AK=KR=RL=LM=MP$.	1. By construction.
2. AS, KZ, RH, LO, MV are $\parallel$ to PB.	2. By construction, § 99.
3. $AZ=ZH=HO=OV=VB$.	3. § 144.

QUERY 1. Why is it desirable to make $\angle BAG$ in the above figure acute rather than obtuse?

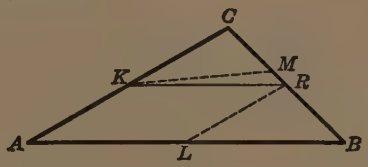
QUERY 2. Why is it desirable to make AK approximately one fifth of AB?

QUERY 3. How would one proceed to divide the line AB into three equal parts?

QUERY 4. Could a line be bisected by this method as conveniently as by the method of § 135?

Theorem 30

147. *The line which joins the mid-points of two sides of a triangle is parallel to the third side and equal to one half of it.*



Given $\triangle ABC$, in which the line KR joins the mid-points of AC and BC.

To prove that KR *is* $\parallel$ *to* AB *and that* $KR = \frac{1}{2} AB$.

Method. Indirect. Draw KM parallel to AB, and prove that it bisects BC and hence coincides with KR.

Proof

1. Draw $KM \parallel$ to AB and, if possible, distinct from KR.	1. § 82.
2. M is the mid-point of CB.	2. § 145.
3. $\therefore$ KM and KR cannot be distinct.	3. § 51, Postulate 1.
4. $\therefore$ KR is $\parallel$ to AB.	4. Steps 1 and 3.
5. Draw $RL \parallel$ to CA.	5. § 82.
6. L is the mid-point of AB, and $AL = BL = \frac{1}{2} AB$.	6. § 145.
7. $ALRK$ is a $\square$.	7. § 121.
8. $KR = AL$.	8. § 126.
9. $KR = \frac{1}{2} AB$.	9. § 50, Axiom 1.

QUERY. What is the relation between the sum of the three sides of $\triangle ABC$ and that of the three sides of $\triangle RKL$?

EXERCISES

1. If K is any point on AB of $\triangle ABC$ and R is the mid-point of AK, L the mid-point of AC, and M the mid-point of CK, then $KRLM$ is a parallelogram. Prove.

2. The lines joining the mid-points of the sides of a triangle divide it into four congruent triangles. Prove.

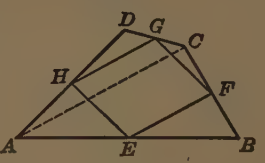
3. The line bisecting two sides of a triangle bisects all lines drawn to the third side from the opposite vertex.

4. The line joining the mid-points of two adjacent sides of a quadrilateral is equal to one half of one of the diagonals of the quadrilateral. Prove.

5. If lines be drawn from any vertex of a parallelogram to the mid-points of the opposite sides, they will divide the diagonal which they intersect into three equal parts.

6. The lines joining the mid-points of the adjacent sides of a quadrilateral form a parallelogram.

HINT. What relation exists between AC and EF?



7. The lines joining the mid-points of the opposite sides of a quadrilateral bisect each other. Prove.

148. Isosceles trapezoid. An *isosceles trapezoid* is one which has the two nonparallel sides equal.

EXERCISES

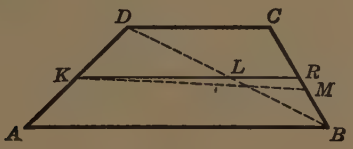
1. The two angles which are adjacent to either base of an isosceles trapezoid are equal.

2. Prove the diagonals of an isosceles trapezoid equal.

3. The opposite angles of an isosceles trapezoid are supplementary. Prove.

Theorem 31

149. *The line joining the mid-points of the non-parallel sides of a trapezoid is parallel to the bases and equal to half their sum.*



Given the trapezoid *ABCD* with *KR* joining the mid-points of the nonparallel sides *AD* and *CB*.

To prove that KR is ∥ to AB and DC and equal to $\frac{1}{2}(AB+DC)$.

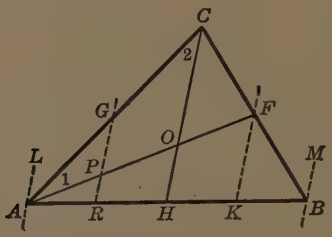
Method. Indirect. Draw KM parallel to AB, and prove that it bisects the diagonal BD and the side BC and therefore coincides with KR. Hence it equals half of AB plus half of DC and is parallel to each.

Proof

1. Draw $KM \parallel$ to AB.	1. § 82.
2. $CM = MB$.	2. § 144.
3. $\therefore$ M and R coincide.	3. By hypothesis and step 2.
4. KR coincides with KM.	4. § 51, Postulate 1.
5. $\therefore$ KR is $\parallel$ to AB.	5. Steps 1 and 4.
6. Then KR is $\parallel$ to DC.	6. § 84.
7. Draw the diagonal BD, cutting KR at L.	7. § 51, Postulate 1.
8. L is the mid-point of BD.	8. § 145.
9. $KL = \frac{1}{2} AB$.	9. § 147.
10. $LR = \frac{1}{2} DC$.	10. § 147.
11. $KL + LR = \frac{1}{2} AB + \frac{1}{2} DC$.	11. § 50, Axiom 3.
12. $KR = \frac{1}{2}(AB + DC)$.	12. Step 11.

Theorem 32

150. *The medians of a triangle meet at a point which is one third of the distance from the mid-point of each base to the opposite vertex.*



Given $\triangle ABC$, **with medians** AF **and** CH, **and** G **the mid-point of** AC.

To prove that

(*a*) AF *and* CH *intersect at some point* O.

(*b*) $AF = 3\ OF$ *and* $CH = 3\ OH$.

(*c*) *The third median,* BG, *passes through* O, *making* $BG = 3\ OG$.

Method. (*a*) Prove that AF is not parallel to CH and hence must intersect it.

(*b*) Divide AB into four equal parts, draw lines through A, R, K, and B parallel to CH, and prove that they divide AF into three equal parts. In a similar manner prove that CH is divided into three equal parts.

(*c*) Prove that BG and AF would also intersect in point O.

Proof

(*a*) 1.	$\angle 1 + \angle 2$ cannot equal two rt. $\measuredangle$.	1. § 100.
2.	$\therefore$ AF is not $\parallel$ to CH, and intersects it.	2. If AF were $\parallel$ to CH, $\angle 1 + \angle 2$ would equal 2 rt. $\measuredangle$. (§ 93.)

(*b*) 1. Draw through points A, G, F, and B lines LA, GR, FK, and $MB \parallel$ to CH.	1. § 99.
2. Since $AG=GC$, $AR=RH$. In like manner, $BK=KH$.	2. By hypothesis and § 145.
3. $AH = HB$.	3. By hypothesis.
4. $AR = RH = HK = KB$.	4. § 50, Axiom 6.
5. $\therefore AP = PO = OF$. Then $AF = 3\,OF$.	5. § 144.

Similarly, by drawing $\parallel$s to AF through C, G, H, and B, it can be proved that $CH = 3\,OH$.

(*c*) This proof shows that O is a point such that $OH = \frac{1}{3}\,CH$ and $OF = \frac{1}{3}\,AF$. In the same way BG and AF can be proved to intersect in a point Q such that $FQ = \frac{1}{3}\,AF$ and $GQ = \frac{1}{3}\,BG$. Hence point Q would coincide with O and the three medians of a triangle would intersect in that point.

QUERY 1. What is the difference between the altitudes of an equilateral triangle, its medians, and the bisectors of its angles?

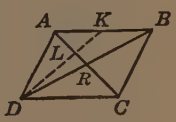
QUERY 2. In an equilateral triangle what four sets of lines intersect in the same point?

EXERCISES ON MEDIANS AND MID-POINTS OF TRIANGLES

1. Prove that if a median of a triangle is perpendicular to one side, the triangle is isosceles.

2. Prove that if two medians of a triangle are equal, the triangle is isosceles.

3. $ABCD$ is a parallelogram. The diagonals intersect in R. The line DK bisects AB at K and cuts AC in L.



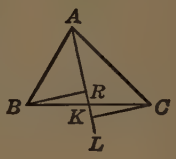
Prove $DL = 2\,LK$ and $LR = \frac{1}{6}\,AC$.

4. Prove that the six triangles into which an equilateral triangle is divided by its three medians are congruent.

5. In a right triangle the median from the vertex of the right angle is one half the hypotenuse. Prove.

6. State the converse of the theorem of the preceding exercise and, if it is true prove it.

7. K is the middle point of the side BC of $\triangle ABC$, and BR and CL are perpendiculars from B and C respectively to AK, produced if necessary.

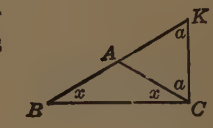


Prove $BR = CL$.

8. The side BC of $\triangle ABC$ is produced to the point K. The $\angle ACB$ is bisected by the line CR, which meets AB at R. A line is drawn through R parallel to BC, meeting AC at L and the line bisecting the exterior angle ACK at G.

Prove $RL = LG$.

9. BC is the base of the isosceles triangle ABC. The side BA is produced its own length to K, and KC is drawn.



Prove that $\triangle BCK$ is a right triangle.

HINT. Consider the sum of the angles of $\triangle BCK$.

10. In $\triangle ABC$ the line KR, parallel to AB, joins K on AC to R on BC and is equal to two fifths of AB.

Prove $AK = \frac{3}{5}\, AC$.

151. Names of polygons. Polygons are named from the number of sides. The more frequently used names follow.

NUMBER OF SIDES	NAME OF POLYGON	NUMBER OF SIDES	NAME OF POLYGON
3	Triangle	8	Octagon
4	Quadrilateral	9	Nonagon
5	Pentagon	10	Decagon
6	Hexagon	12	Dodecagon
7	Heptagon	n	n-gon

152. Equilateral polygon. A polygon is *equilateral* if all its sides are equal.

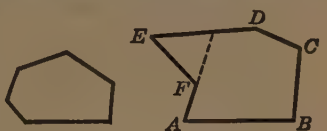
153. Equiangular polygon. A polygon is *equiangular* if all its angles are equal.

154. Regular polygon. A *regular* polygon is a polygon all of whose angles are equal and all of whose sides are equal. Such a polygon is both equiangular and equilateral.

Equilateral triangle | Square | Regular pentagon | Regular hexagon | Regular octagon

155. Convex and concave polygons. A polygon is called *convex* if each of its angles is less than two right angles, and *concave* if any one of its angles is greater than two right angles.

No straight line can cut a convex polygon in more than two points, but it may cut a concave polygon in four or more points.



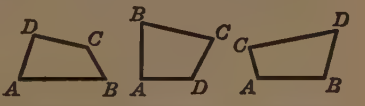
Convex polygon | Concave polygon

Polygon *ABCDEF* is concave, as the interior angle at *F* is greater than a straight angle.

Only convex polygons are studied in this book.

156. Diagonal. A *diagonal* of a polygon is a line joining any two nonconsecutive vertices.

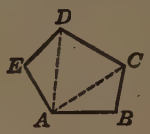
157. Order in naming the vertices of a polygon. In stating exercises in which it is necessary to name a polygon of four or more sides by the letters at its vertices, it is customary to name the vertices in succession. Thus *ABCD* is correctly lettered in the first two figures but not in the third.



Correct | Correct | Incorrect

NUMERIC EXERCISES

1. If the opposite angles of a quadrilateral are equal, the adjacent angles are supplementary.



HINT. $2a + 2b = ?$ Etc.

2. Find the sum of the angles of a pentagon.

HINT. From one vertex draw all the diagonals possible. Then find the sum of all the angles of the triangles so formed.

3. Find the number of degrees in each angle of a regular pentagon.

4. In a regular pentagon, find the number of degrees in the angle between two diagonals from the same vertex.

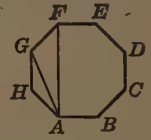
5. Find the sum of the angles of a hexagon.

6. Find the number of degrees in each angle of an equiangular hexagon.

7. Find the sum of the angles of an octagon.

8. Find the number of degrees in each angle of a regular octagon.

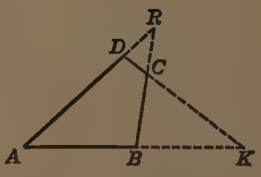
9. The adjacent octagon is regular. Find the number of degrees in each angle of $\triangle AHG$ and in $\angle AGF$.



10. Find the number of degrees in each angle of an equiangular decagon.

11. The sides AB and DC of quadrilateral $ABCD$ meet when produced in point K, and AD and BC produced meet in R.

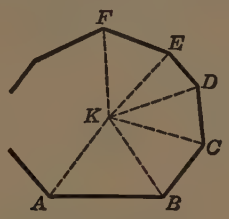
Prove $\angle RCK = \angle R + \angle K + \angle A$.



12. Find the angle between an adjacent pair of diagonals of a regular hexagon.

Theorem 33

158. *If n is the number of sides of a convex polygon and s is the sum of its interior angles, then $s = (n - 2)$ straight angles.*



Given any convex polygon $ABCDEF \cdots$ having n sides.

To prove that the sum of the interior angles of $ABCDEF \cdots$ is $(n - 2)$ st. ∠.

Method. From any point K within the polygon draw lines to the vertices. The sum of the angles of the polygon is the sum of the angles of all the triangles less the two straight angles at K.

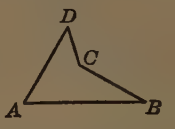
Proof

1. From K, any point within the polygon, draw a line to each vertex forming n △.	1. § 51, Postulate 1.
2. The sum of the ∠ of each △ = 1 st. ∠.	2. § 100.
3. The sum of the ∠ of the n △ = n st. ∠.	3. Steps 1 and 2.
4. The ∠ about the point K = 2 st. ∠.	4. § 44, Example 2.
5. ∴ Sum of interior ∠ of a polygon is $s = (n - 2)$ st. ∠.	5. § 50, Axiom 4.

QUERY. How many degrees are there in the sum of the interior angles of a polygon of n sides?

EXERCISES

1. In the adjacent concave quadrilateral, is the sum of the angles four right angles?



2. If one interior angle of an equiangular polygon of n sides is x right angles, prove that $x = \frac{2n-4}{n}$ right angles.

3. A polygon has 16 sides. Find in degrees the sum of its interior angles.

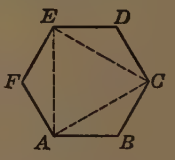
4. The sum of the interior angles of a polygon is 28 right angles. How many sides has it?

HINT. $28 = 2n - 4$ right angles, etc.

5. Find the number of sides of a polygon the sum of whose interior angles is 5400°.

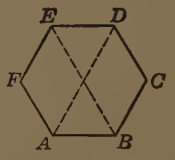
6. How many degrees are there in each interior angle of a regular decagon?

7. One interior angle of a regular polygon contains 172°. How many sides has the polygon?



8. $ABCDEF$ is a regular hexagon. Prove that the lines AC, CE, and EA form an equilateral triangle.

9. Prove that the longer diagonals of a regular hexagon bisect the angles at the vertices which they connect.

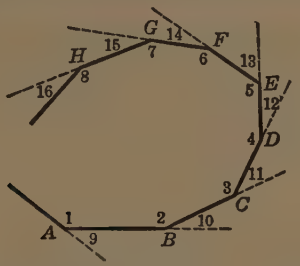


10. $ABCDEF$ is a regular hexagon. Prove that if the diagonals AD and BE are drawn, two equilateral triangles are formed.

11. In parallelogram $ABCD$, $\angle B$ is 120°, and the diagonal BD makes a right angle with AD. If DK is $\perp$ to AB at K, prove that BD is twice DK.

Theorem 34

159. *The sum, S, of the exterior angles of any convex polygon, counting one at each vertex, is four right angles.*



Given the convex polygon $ABCDE\cdots$ **having** n **sides.**

To prove that the sum of the exterior angles, one at each vertex, is 4 rt. ∡.

Method. Find the sum of the interior and exterior angles at each vertex for the n vertices of the polygon. Subtract from this result the sum of the interior angles.

Proof

1. Extend the sides, forming one exterior ∠ at each vertex.	1. § 51, Postulate 3.
2. The sum of the exterior ∠ and the interior ∠ at each vertex is 1 st. ∠.	2. § 31.
3. Hence the sum of the n interior ∡ and the n exterior ∡ is n st. ∡.	3. Step 2.
4. But the sum of the interior ∡ alone is $(n-2)$ st. ∡.	4. § 158.
5. ∴ the sum, S, of the exterior ∡ of any polygon is $S = [n-(n-2)]$ st. ∡, or 4 rt. ∡.	5. Steps 3 and 4.

EXERCISES

1. How many degrees are there in each exterior angle of a regular hexagon? a regular octagon? a regular decagon?

2. Find the number of sides of a polygon if the sum of its exterior angles equals the sum of its interior angles.

HINT. Let n be the number of sides of the polygon. Then, by §§ 158 and 159, $4 = 2n - 4$.

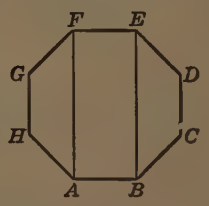
3. Find the number of sides of a polygon if the sum of its exterior angles is twice the sum of its interior angles.

4. Find the number of sides of a polygon if the sum of its interior angles is twice the sum of its exterior angles.

5. Does a polygon exist the sum of whose angles is 1874°?

6. Find the number of sides of a polygon if the sum of its interior angles exceeds the sum of its exterior angles by 34 right angles.

7. Find the number of sides of a polygon if the sum of its interior angles exceeds the sum of its exterior angles by 1260°.



8. $ABCDEFGH$ is a regular octagon. Prove that $AFGH$ is an isosceles trapezoid and that $ABEF$ is a rectangle.

160. Inequalities. In a number of theorems now to be proved, inequalities occur.

The signs of inequality are $>$, which is read "is greater than," and $<$, which is read "is less than."

161. Order of inequalities. Two inequalities are said to exist in the same order if the left member in each is the greater member or the less member.

Thus, $9 > 7$ and $6 > 3$ exist in the same order; also $x < m$ and $c < a$ exist in the same order.

Two inequalities are said to be in the reverse order if the left member of one is its greater member and the right member of the other is its greater member, or vice versa.

Thus, the inequalities $7 > 3$ and $8 < 10$ are in reverse order.

162. Axioms on inequalities. The axioms stated in § 50 apply to equal numbers, that is, to equations. In handling inequalities, either alone or in connection with equations, a number of additional axioms are needed. The following axioms cover most of the cases which arise:

Axiom 8. The whole is greater than any of its parts.

Axiom 9. If the first of three numbers is greater than the second and the second is greater than the third, the first is greater than the third.

This axiom states that if $a > b$ and $b > c$, then $a > c$.

Axiom 10. If the same number, positive or negative, is added to or subtracted from each member of an inequality, the results are unequal in the same order.

Thus, adding 5 to each member of the inequality $11 > 8$ gives $16 > 13$. Adding -5 gives $6 > 3$.

Axiom 11. If both members of an inequality are multiplied or divided by the same positive number, the results are unequal in the same order.

Thus, multiplying both members of the inequality $9 > 7$ by 3 gives $27 > 21$.

Axiom 12. If the corresponding members of two or more inequalities which are in the same order are added, the sums are unequal in the same order.

Applying this axiom to $6 > 4$ and $11 > 7$ gives $17 > 11$.
In general, if $a > x$ and $b > y$, then $a + b > x + y$.

Axiom 13. If unequals are subtracted from equals, the results are unequal in the reverse order.

Subtracting $8 > 5$ from $12 = 12$ gives $4 < 7$.

EXERCISES

1. If the members of an equation are subtracted from the corresponding members of an inequality, in what order is the resulting inequality?

2. If the members of an inequality are subtracted from the corresponding members of an equation, in what order is the resulting inequality?

3. Is Axiom 11 true if the word "positive" is changed to "negative"?

4. Write down a numeric inequality like $9 > 5$ and test the truth of Axioms 8 to 13 inclusive.

5. If $x + 3 > 5$, show that $x > 2$.

6. If $2x - 8 > 10$, show that $x > 9$.

7. If $\frac{3x}{5} - 6 < 12$, show that $x < 30$.

8. If $4x + K > 25$ and $K = -7$, show that $x > 8$.

9. If $x + 10 > 30$ and $x - y = 4$, show that $y > 16$.

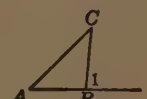
10. If $2x + 3 > 10$ and $x + y = 5$, show that $y < \frac{3}{2}$.

11. Show that $\frac{a}{b} + \frac{b}{a} > 2$ for all values of a and b which are unequal.

HINT. For all such values of a and b, $a - b \neq 0$. Consequently $a^2 - 2ab + b^2 > 0$, etc.

12. Does it follow from Exercise 11 that any number added to its reciprocal is greater than 2?

163. Fundamental inequalities for triangles. (*a*) An exterior angle of a triangle is greater than either opposite interior angle.



By § 102, $\angle 1 = \angle A + \angle C$. Therefore $\angle 1 > \angle A$ or $\angle C$, by Axiom 8, § 162.

(*b*) Any side of a triangle is less than the sum of the other two sides. (§ 51, Postulate 2.)

(*c*) Any side of a triangle is greater than the difference of the other two sides.

From $AB + BC > AC$, we have, by Axiom 10, $BC > AC - AB$.

(*d*) A straight line is the shortest line between two points.

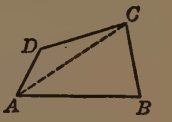
EXERCISES

1. $ABCD$ is a quadrilateral. Prove that the perimeter of $\triangle ABC$ is less than that of the quadrilateral.

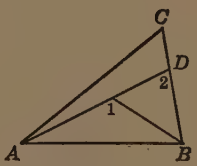
2. $ABCDE$ is a pentagon. Draw BE and prove that the perimeter of $BCDE$ is less than that of the pentagon.

3. Any side of a quadrilateral is less than the sum of the other three sides. Prove.

HINT. $AD + DC > AC$ and $AC + BC > AB$. Add these inequalities and simplify.



4. In the accompanying figure, prove $\angle 1 > \angle C$.



5. In $\triangle ABC$, $AC = BC$. BL is drawn to any point on AC.

Prove $\angle BLC > \angle ABC$.

6. In the equilateral triangle ABC the line CL is drawn to L on AB so that $AL < BL$.

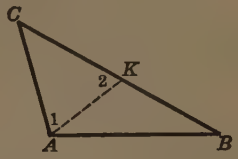
Prove $\angle ALC \neq$ a right angle.

7. The perimeter of a parallelogram is greater than the sum of its diagonals. Prove.

HINT. $a + d > x + x$, and $a + b > m + m$.
$\therefore 2a + b + d > 2x + 2m$. Etc.

Theorem 35

164. *If two sides of a triangle are unequal, the angles opposite them are unequal and the greater angle lies opposite the greater side.*



Given $\triangle ABC$, **in which** $CB > CA$.

To prove that $\angle CAB > \angle B$.

Method. On the greater side, CB, lay off CK equal to CA, draw AK, and prove that $\angle 1 = \angle 2$. Then show that $\angle CAB > \angle 1$ and hence $\angle 2 > \angle B$.

Proof

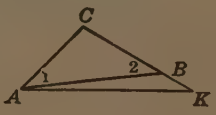
1. On CB lay off CK equal to CA, and draw AK.	1. § 51, Postulate 1.
2. K lies between C and B.	2. $CB > CA$, by hypothesis.
3. $\angle CAB > \angle 1$.	3. § 162, Axiom 8.
4. $CK = CA$.	4. By construction.
5. $\angle 1 = \angle 2$.	5. § 67.
6. $\angle 2 > \angle B$.	6. § 163 (a).
7. $\angle 1 > \angle B$.	7. § 50, Axiom 7.
8. $\therefore \angle CAB > \angle B$.	8. § 162, Axiom 9.

EXERCISES

1. In $\triangle ABC$, side $AB = 10$, $BC = 12$, and $CA = 15$. Which is the largest angle? the smallest?

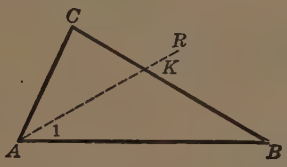
2. A triangle with unequal sides has unequal angles.

3. AC and BC are equal sides of $\triangle ABC$. CB is extended to K, and AK is drawn. Prove $\angle CAK > \angle K$.



Theorem 36

165. *If two angles of a triangle are unequal, the sides opposite them are unequal and the greater side lies opposite the greater angle.*



Given $\triangle ABC$, **in which** $\angle CAB > \angle B$.

To prove that $BC > AC$.

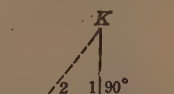
Method. From the vertex of the greater angle, A, draw AR, making $\angle 1 = \angle B$. Prove that AR intersects CB in K, $AK = BK$, and $BK + KC = AK + KC > AC$.

Proof

1. Draw AR so that $\angle RAB = \angle B$.	1. § 85.
2. $\angle CAB > \angle B$.	2. By hypothesis.
3. $\angle CAB > \angle 1$.	3. § 50, Axiom 7.
4. Hence AR is not $\parallel$ to BC and must intersect BC in some point K.	4. If they were $\parallel$, $\angle 1 + \angle B$ would equal 180° (§ 93). This is impossible (§ 100).
5. Now in $\triangle ABK$, $AK = BK$.	5. § 107.
6. But $AK + KC > AC$.	6. § 163 (*b*).
7. $\therefore BK + KC = BC > AC$.	7. § 50, Axiom 7.

166. ***Corollary.*** *The perpendicular from a point outside a straight line is the shortest line from the point to the line.*

HINT. Compare $\angle 1$ with $\angle 2$.

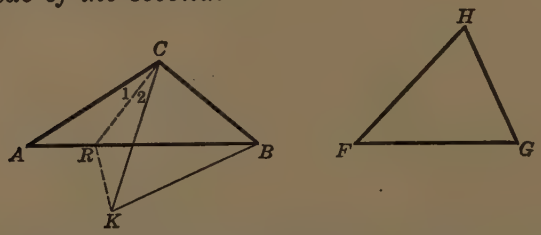


QUERY 1. Which is the greatest side of a right triangle? of an obtuse triangle?

QUERY 2. In $\triangle ABC$, $\angle A = 48°$ and $\angle B = 56°$. Which is the greatest side of the triangle? the least?

Theorem 37

167. *If two triangles have two sides of one equal respectively to two sides of the other and the included angle of the first greater than the included angle of the second, the third side of the first is greater than the third side of the second.*



Given $\triangle ABC$ **and** FGH, **in which** $AC = FH$, $BC = GH$, **and** $\angle ACB > \angle H$.

To prove that $AB > FG$.

Method. Superpose $\triangle FGH$ upon $\triangle ABC$ so that the smaller angle falls upon the larger and GH coincides with BC, point F falling at K. Then draw CR bisecting $\angle ACK$, join R and K, and prove $\triangle CRK \cong \triangle CRA$. Hence $RA = RK$ and $BA = BR + RA = BR + RK$. Hence $BA > BK$.

Proof

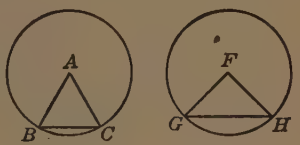
1. Superpose $\triangle FGH$ upon $\triangle ABC$ so that the side GH will fall on its equal, BC, the point G falling at B, the point H at C, and the point F at some point K.	1. § 51, Postulate 7.
2. Line HF will fall within $\angle ACB$ in the position CK.	2. $\angle H < \angle ACB$, by hypothesis.

3. Let the line CR bisect $\angle ACK$ and intersect AB at R.	3. § 34.
4. Draw RK.	4. § 51, Postulate 1.
5. In $\triangle ACR$ and KCR, $CR = CR$.	5. Identical.
6. $\angle 1 = \angle 2$.	6. By construction.
7. $AC = HF$, or its equal, CK.	7. By hypothesis.
8. $\triangle ACR \cong \triangle KCR$.	8. § 62.
9. $RA = RK$.	9. § 64.
10. $BR + RK > BK$.	10. § 163 (b).
11. $BR + RA > BK$.	11. § 50, Axiom 7.
12. $AB > FG$.	12. Step 11.

EXERCISES

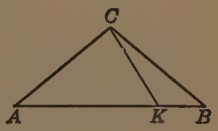
1. In the adjacent figure A and F are the centers of equal circles and $\angle F > \angle A$.

Prove $BC < GH$.



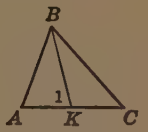
2. In the adjacent figure, $AC = BC$, and K is any point on AB.

Prove $CK < AC$ or BC.



3. In the adjacent figure, $\angle 1$ is acute and $AK = KC$.

Prove $BC > BA$.

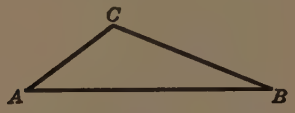


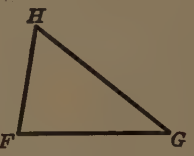
4. If a parallelogram is not a rectangle, its diagonals are unequal.

HINT. Let $\angle A$ be less than $\angle B$. Then compare $\triangle ABD$ and ABC.

Theorem 38

168. *If two triangles have two sides of one equal respectively to two sides of the other and the third side of the first greater than the third side of the second, the included angle of the first is greater than the included angle of the second.*





Given $\triangle ABC$ and FGH, in which $AC = FH$, $BC = GH$, and $AB > FG$.

To prove that $\angle C > \angle H$.

Method. Indirect. Three and only three possible relations between $\angle C$ and $\angle H$ exist: (*a*) $\angle C < \angle H$; (*b*) $\angle C = \angle H$; (*c*) $\angle C > \angle H$. Prove the first two impossible. Then the third must be true.

Proof

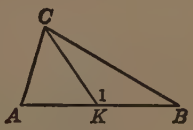
1. Let us suppose that either (*a*) $\angle C < \angle H$; (*b*) $\angle C = \angle H$; (*c*) $\angle C > \angle H$.	1. These possibilities and no others exist.
2. If we suppose (*a*) is true, then it follows that $AB < FG$.	2. § 167.
3. But $AB > FG$.	3. By hypothesis.
4. Hence (*a*) is not true.	4. It leads to a false conclusion.
5. If we suppose (*b*) is true, then $\triangle ABC \cong \triangle FGH$.	5. § 62.
6. Then $AB = FG$.	6. § 64.

7. But $AB > FG$.	7. By hypothesis.
8. Hence (b) is not true.	8. It leads to a false conclusion.
9. Now one of the statements (a), (b), or (c) must be true.	9. There are no further possibilities.
10. Hence (c) is true.	10. (a) and (b) have been proved untrue.

EXERCISES

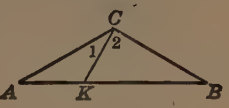
1. $BC > AC$ in $\triangle ABC$, and CK is the median to side AB.

Prove that $\angle 1$ is an obtuse angle.



2. Point K is taken in the base AB of isosceles triangle ABC so that $AK < BK$.

Prove $\angle 2 > \angle 1$.



169. Proof by reductio ad absurdum. The proof of the preceding theorem is an example of a method of proof which has long had the Latin name *reductio ad absurdum*. Frequently such a proof is called *indirect*.

The two names, however, are not interchangeable. For example, in Theorem 30 line KR was proved parallel to AB by showing that it coincided with KM. This proof is indirect but is not a proof by reductio ad absurdum.

One essential feature of proof by reductio ad absurdum consists in making all the assumptions possible under the hypothesis (usually three) and then disproving the truth of all but one. Another essential feature of the method is the manner of disproving each of the false assumptions made. This consists in reasoning from the assumption to a conclusion which is seen to be false because it contradicts some statement of the hypothesis, some axiom, some obvious fact, or some previously established theorem. Of

course this result disproves the truth of the assumption, for if correct reasoning leads from an assumption to a conclusion which is untrue, then the assumption on which the reasoning is based is untrue.

Such a method is conclusive, if we make all the assumptions, both true and false, which are possible under the given hypothesis. Of these we must know beforehand that one must be true and only one can be true. If, then, we prove all the assumptions false but one, that one must be true.

QUERY 1. What theorems, if any, before Theorem 38 are proved by reductio ad absurdum?

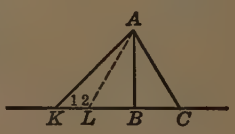
QUERY 2. What theorems of Book I have been proved indirectly?

QUERY 3. Cite three theorems of Book I which have been proved directly.

MISCELLANEOUS EXERCISES ON INEQUALITIES

1. If the diagonals of a parallelogram are unequal, it is not a rectangle. Prove.

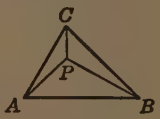
2. If two line segments drawn from a point in a perpendicular to a straight line cut off unequal distances from the foot of the perpendicular, the one cutting off the greater distance is the greater. Prove.



HINT. Take $BL = BC$, then compare $\angle 1$ with $\angle 2$ and AL with AK and AC. Etc.

3. If the lines AB and CD intersect, then the sum of AB and CD is greater than the sum of AC and DB.

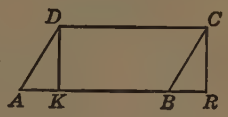
4. If from a point within a triangle lines are drawn to the vertices, their sum is greater than half the sum of the sides of the triangle.



HINT. Write two other inequalities like $PA + PB > AB$ and combine all three.

5. A rectangle and a parallelogram have the same base and equal altitudes. Which has the greater perimeter?

Prove it.



HINT. Let $ABCD$ be the parallelogram and $KRCD$ the rectangle.

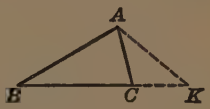
6. In $\triangle ABC$, K is any point in AB, and R any point in BC.

Prove $AB + BC > AK + KR + RC$.

7. If one side of a triangle is divided into two parts by a perpendicular from the opposite vertex, each part of that side is less than the adjacent side of the triangle.

8. AB and BC are equal sides of $\triangle ABC$. BC is produced to the point K and KA is drawn.

Prove $\angle BAK > \angle BKA$.



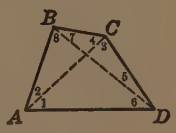
9. Any point (except the vertex) in one of the equal sides of an isosceles triangle is unequally distant from the extremities of the base. Prove.

10. ABC is a triangle in which KB and KC bisect the $\measuredangle B$ and C respectively, and $AB > AC$.

Prove $KB > KC$.

11. In the quadrilateral $ABCD$, AD is the longest and BC the shortest side.

Prove $\angle B > \angle D$ and $\angle C > \angle A$.



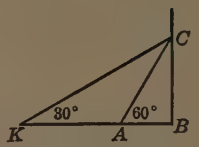
12. In $\triangle ABC$, $\angle A$ is bisected by a line meeting BC at K.

Prove $AB > BK$ and $AC > CK$.

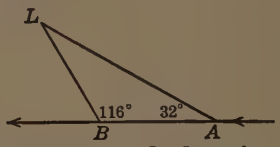
HINT. Compare the angles at K with the two at A.

APPLICATIONS

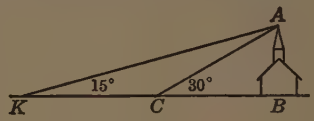
1. BC is a road at right angles with the road KB. A church is at C. It is found that $\angle CAB = 60°$ and $\angle K = 30°$. If AK is $2\frac{1}{2}$ mi., find AB.



2. A ship is sailing west at 12 mi. per hour. At 2 P.M. a lighthouse L is in such a position that $\angle A = 32°$. At 4.30 P.M. $\angle LBA = 116°$. How far from L is the ship when at B?

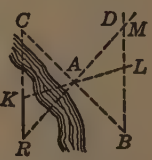


3. To determine the height of a tower, a man finds point C where $\angle ACB = 30°$. Then he walks away from the tower to a point K where $\angle K = 15°$. The distance KC is found to be 100 yd. If KCB is a horizontal line, find AB.

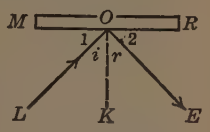


4. A tunnel is being dug through a hill H in the direction AB. In order to work from the other side also, $\angle RAC$ is made equal to 150°, $AC = 1\frac{1}{2}$ mi., and $\angle C = 60°$. At what point on CK and in what direction must the tunnel be dug so that the two parts will form a straight line?

5. The distance between two points, K and R, across a river can be determined without crossing the river as follows: Select point C in line with RK. Then choose point B and locate A, the mid-point of BC. Lay off BD, making $\angle B = \angle C$. Then by sighting, locate on line BD point M in line with A and R and line L in line with A and K. Measurement of what line gives the length of KR? Prove.



6. In the adjacent figure, MR is a mirror and OK is a perpendicular to it at O. Light from L striking the mirror at O is reflected in the direction OE so that $\angle i = \angle r$. This law is usually stated thus: The angle of incidence ($\angle i$) equals the angle of reflection ($\angle r$).

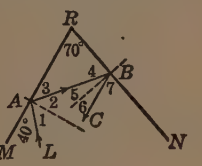


Prove that $\angle 1 = \angle 2$ also.

7. The image of an object A seen in a plane mirror by the eye at E appears to be at A', as far behind the mirror MR as A is in front of it and so that AA' is $\perp$ to MR. That is, rays of light from A strike the mirror at L and are reflected to E. Show that $\angle ALR = \angle ELM$.

8. Referring to the figure of the preceding exercise, suppose that the ray of light from A to the mirror was reflected to E from some other point than L. Show that the distance the light travels would be longer than $AL + LE$.

9. In the adjacent figure, MR and RN are two mirrors. A ray of light reflected from the one mirror at A to the other at B follows the path $LABC$. From the two given angles determine the number of degrees in each of the angles 1 to 7 inclusive.



REVIEW QUESTIONS

1. What is (*a*) an isosceles trapezoid? (*b*) a median of a triangle? (*c*) a median of a trapezoid?

2. Is the line joining the mid-points of two opposite sides of a parallelogram equal to either of the other two sides? Is there a theorem proved in the text of which the preceding is a special case? If so, state the theorem.

3. Do the altitudes of an obtuse triangle intersect in a common point? If so, where is the point?

4. What four systems of lines in a triangle have common points of intersection?

5. Is an equilateral polygon regular? Is a rhombus a regular polygon?

6. Is an equiangular polygon regular? Are all rectangles regular? Is any rectangle regular?

7. What are the respective names of polygons having 5, 6, 8, and 10 sides respectively?

8. What is the difference between a convex and a concave polygon?

9. What is the sum of the exterior angles of any polygon? the sum of the interior angles?

10. How many sides has a polygon if the sums of its interior and of its exterior angles are equal?

11. What is the difference between producing line KR to P and producing line RK to P?

12. State two facts concerning the line joining the midpoints of two sides of a triangle.

13. State two facts concerning the line joining the midpoints of the two nonparallel sides of a trapezoid.

14. Is any axiom for equalities not true for inequalities?

15. What is the result of dividing $18 > 12$ by -3?

16. What is the result of transposing all the terms of the left member of $a^2 - 2ab + b^2 > 0$ to the right member?

17. State three conditions which make any two triangles congruent.

18. State three conditions which make two right triangles congruent.

19. What relation exists between two sides of a 30°-60° right triangle?

20. What is the converse relation of 19? Is it true?

21. What set of lines in a triangle divide one another into segments in a definite way?

22. Classify triangles in two different ways.

23. What is the method most frequently used of proving two lines equal or two angles equal?

24. In what other ways are angles proved equal? lines?

25. What is meant by a direct proof? Give an example.

26. What is meant by an indirect proof? Give an example.

Summary

(Black figures refer to sections)

170. Résumé of Introduction and Book I. The Introduction and Book I deal with many facts and ideas which are new to the student. These come, however, in logical order, and consequently the fundamental concepts may be grouped under a few heads.

1. *Geometric figures studied.* Point, **26**; straight line, **27**; angle, **28**; plane, **29**; kinds of angles, **31, 36, 40, 41**; triangle, **53**; kinds of triangles, **54–56**; polygons, **118–124**; perpendiculars, **35**; parallel lines, **81.**

2. *Terms used.* Assumption, **19**; axiom, **23**; postulate, **23**; proposition, **42**; theorem, **43**; corollary, **68**; proof, **46**; hypothesis, **45**; conclusion, **45**; converse, **95**; hypotenuse, **57**; degree, **37**; complement, **38**; supplement, **39**; alternate interior angles, **90**; alternate exterior angles, **90**; exterior angles, **101**; vertical angles, **47**; adjacent angles, **33**; inequalities, **160, 162.**

3. *Assumptions.* Axioms, 23, 50, 162; postulates, 23, 51.

4. Theorems on congruent triangles, 62, 66, 71.

5. Theorems on congruent right triangles, 88, 105, 109

6. Theorems on perpendicular lines, 78, 83.

7. Theorems on parallel lines, 84, 86, 87, 91, 92, 93, 94, 96, 97, 98.

8. Theorems on parallelograms, 125, 126, 127, 128, 129, 130, 131, 132, 133.

9. Theorems on isosceles and equilateral triangles, and right triangles with one angle 30°, 67, 69, 70, 107, 108, 110, 111.

10. Theorems on sum of angles of polygons, 100, 158, 159.

11. Lines in a triangle which intersect in a common point, 138, 139, 143, 150.

A WISE MAN OF ANCIENT GREECE

Thales of Miletus (640–546 B.C.) was a man who had varied interests. He was at different times in his life a merchant, a statesman, a philosopher, a scientist, and a mathematician. Thales was the first of the famous Seven Wise Men of Greece and is the only one who was a scientific man. His business affairs took him for a time to Egypt, where he learned what the Egyptians knew of science and geometry. He seems to have been the first man to appreciate the necessity for a scientific proof. He is given credit for the following theorems of this text:

1. *If two lines intersect, the vertical angles are equal* (§ 48).

2. *The angles at the base of an isosceles triangle are equal* (§ 67).

3. *Two triangles are congruent if they have a side and two adjacent angles respectively equal* (§ 66).

4. *A circle is bisected by its diameter* (§ 171, 6).

5. *An angle inscribed in a semicircle is a right angle* (§ 230).

6. *The sides of similar triangles are proportional* (§ 272).

Thales applied the third theorem above to the measurement of the distances of ships from shore, and he astonished the king

of Egypt by using the sixth one to determine the heights of the pyramids from the lengths of their shadows.

Study of astronomy by the Chaldeans had enabled them to predict eclipses even before the time of Thales. One of his most striking achievements, however, was the prediction of an eclipse believed to have been that of May 28, 585 B.C. The common belief was that such events were due to the whims of the gods. Thales boldly declared that they were the result of the fact that the heavenly bodies moved according to fixed laws. Thales taught that the year had 365 days and that the moon received its light from the sun, and seems to have been the first of the Greeks to teach that the earth was a sphere.

It was Thales, also, who first noticed that a piece of amber when rubbed becomes electrified. Moreover, he expressed the conviction that there is some unifying principle which links together all the physical phenomena and is able to make them intelligible, and that all matter is made of one primordial element, the search for which should be the aim of natural science. Modern physics has shown that an intimate relation exists between heat, light, electricity, and magnetism, and it knows that electrons are a constituent of the atoms of all substances. Thus we see that, in addition to doing fundamental work in geometry, Thales was the first to describe the spirit which guided the growth of physical science in all ages and to express his belief in the ultimate nature of matter, which the very latest work in physics comes near confirming. These facts constitute a very remarkable tribute to the genius of the man.

In the Middle Ages the Gothic arch was used in cathedrals because of its beauty

Courtesy of the Metropolitan Museum of Art, New York

BOOK II

CIRCLES

The method of making no mistake is sought by everyone. The logicians profess to show the way, but the geometers alone ever reach it, and aside from their science there is no genuine demonstration. — PASCAL

171. Preliminary assumptions. From the definitions of the circle given in §§ 8–11 certain important facts immediately follow:

1. All radii of the same circle or of congruent circles are equal.

2. Circles having equal radii are congruent.

3. If the centers of congruent circles coincide, the circles coincide, and conversely.

4. A diameter is equal to the sum of two radii.

5. A point is within, on, or outside a circle according as its distance from the center is less than, equal to, or greater than the radius.

6. A diameter bisects the circle and the area of the surface it bounds.

7. If a line bisects a circle, it is a diameter.

8. A straight line can intersect a circle in only two points.

As in the case of angles, the terms *equal* and *congruent* are used interchangeably. If two circles are congruent, they are equal; if they are equal, they are also congruent.

These statements will be taken as postulates, though they might be presented as simple theorems to be proved. The solution of the exercises on page 148 will make clear the truth of these postulates without formal proof.

EXERCISES

1. Give the definition of (*a*) circle, (*b*) radius, (*c*) diameter, (*d*) arc.

2. What bearing has the definition of a circle on assumption 1?

3. If two circles with equal radii be superposed so that their centers coincide, will the circles coincide? Why? What bearing have these answers on assumptions 2 and 3?

4. Why is the diameter of a circle twice its radius?

5. Draw a circle with center O whose radius is 2 in. Where is a point whose distance from O is less than 2 in.? greater than 2 in.?

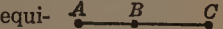
6. Suppose arc $ALMB$ to be revolved about the diameter AB in an attempt to prove it equal to arc ADB. Will line AB change its position by the motion? Can arc $ALMB$ fall in the position of the dotted line $AL'M'B$? Why? What bearing have these questions on assumption 6?

172. Interior of a circle. The portion of the plane inclosed by a circle is called the *interior of a circle.*

173. Contrast between a straight line and a circle. It has already been stated, in § 6, that elementary geometry deals only with the straight line, the circle, and figures made up of straight lines and circles. A fundamental property of a straight line is stated in this postulate: Through two points one and only one straight line can be drawn. This property was assumed without proof. Does a similar equally important property exist for the circle? It will be shown that the answer to this question is "yes."

QUERY 1. Can a circle pass through three points, A, B, and C, which are on the same straight line AC?

(a) On what line are the points which are equidistant from A and B? (§ 137)



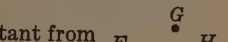
(b) On what line are the points equidistant from B and C?

(c) Do these lines intersect?

(d) What, then, is the answer to Query 1?

QUERY 2. Can a circle pass through three points, F, G, and H, which are not in the same straight line?

(a) On what line are all the points equidistant from F and G?



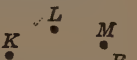
(b) On what line are all the points equidistant from G and H?

(c) Do these lines intersect?

(d) What, then, is the answer to Query 2?

QUERY 3. Can a circle pass through any four points no three of which are in a straight line?

(a) Will the perpendicular bisectors of KL and LM intersect at some point P?



(b) Will the perpendicular bisector of MR always pass through P? May it sometimes do so?

EXERCISES

1. Take three points, F, G, and H, which do not lie in a straight line. Construct the mid-perpendiculars to the line segments FG, GH, and HF. Make your construction accurately. Do the mid-perpendiculars all meet in a point?

2. Take four points at random, but so that they are clearly not on a straight line. Construct the four mid-perpendiculars of the lines obtained by joining the four points. Do these mid-perpendiculars meet in a point?

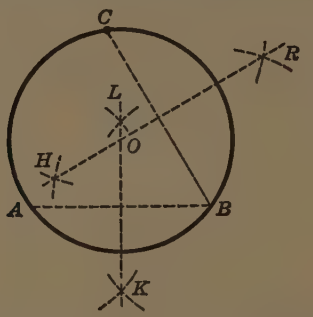
The foregoing queries and constructions suggest the following important property of a circle: Through three points not in the same straight line one circle and only one can pass.

The circle is the simplest curve which can do this.

The property of a circle just stated will now be proved as a theorem.

Theorem 1

174. *Through three points not in the same straight line one circle and only one can pass.*



Given A, B, and C, **three points not in the same straight line.**

To prove that one circle and only one can pass through points A, B, ***and*** C.

Method. Draw the perpendicular bisectors of AB and BC, and prove that they intersect. Then show that the point of intersection is equally distant from A, B, and C.

Proof

1. Draw AB and BC.	1. § 51, Postulate 1.
2. Construct the $\perp$ bisectors of AB and BC.	2. § 135.
3. KL is $\parallel$ to HR or intersects it.	3. Only the two possibilities exist.
4. If KL is $\parallel$ to HR, KL is $\perp$ to BC.	4. § 87. If a line is $\perp$ to one of two $\parallel$s, it is $\perp$ to the other also.
5. If KL is $\perp$ to both BC and AB, then AB is $\parallel$ to BC.	5. § 83. Two lines $\perp$ to the same line are $\parallel$.

6. But this is impossible since AB intersects BC.	6. By hypothesis.
7. $\therefore$ KL intersects HR at some point O.	7. Only possibility remaining.
8. Hence $OA = OB = OC$.	8. § 136. If a point is on the mid-perpendicular of a line it is equally distant from the ends of the line.
9. Hence a $\odot$ having AO as a radius and O as a center will pass through A, B, and C.	9. § 8. Definition of a $\odot$.
10. This is the only $\odot$ which can be passed through points A, B, and C.	10. HR and KL intersect in only one point.
11. Hence one $\odot$ and only one can be drawn through three points which are not in the same straight line.	11. Steps 9 and 10.

EXERCISE

Construct a circle passing through the vertices of a given triangle.

QUERY 1. Can a circle pass through any two points? How many such circles are possible? On what line must the centers lie?

QUERY 2. If three points are in a straight line, can a circle pass through them? Why?

QUERY 3. Can a circle pass through any four points?

QUERY 4. Is the distance of the point of intersection of the diagonals of a rectangle from each of its vertices the same? Why?

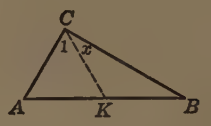
QUERY 5. Can a circle pass through the four vertices of a rectangle? If so, what will be its center and its radius?

QUERY 6. Can a circle pass through the four vertices of any parallelogram? Explain.

QUERY 7. Can a circle be drawn through the vertices of any quadrilateral except a rectangle?

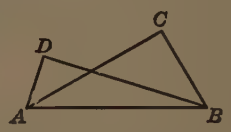
EXERCISES

1. In the right triangle ABC, CK is drawn making $\angle x = \angle B$. Show that $\angle 1 = \angle A$ and that $KC = KA = KB$.



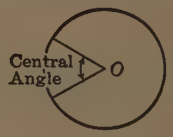
2. Can a circle always be found that will pass through the three vertices of a right triangle? If so, what is its center? its radius?

3. ABC and ABD are two right triangles on the same hypotenuse AB. Show that a circle can pass through A, B, C, and D. What is the center of this circle? What is its radius?



175. Circumference. The length of a circle is called the *circumference*.

The use of the terms *circle* and *circumference* has long been confusing. Both words have been used to designate the curve itself. The word "circle" has been and still is sometimes used to denote both the curved line itself and the portion of the plane bounded thereby. In this text the latter will be called the interior of the circle.



176. Central angle. A *central angle* is an angle whose vertex is at the center and whose sides are radii.

177. Chord. A *chord* is a line segment whose extremities are on the circle.

178. Semicircle. A *semicircle* is an arc which is half the circle.

179. Minor arc. A *minor arc* is less than a semicircle.

180. Major arc. A *major arc* is greater than a semicircle.

Every chord, except a diameter, has two unequal arcs, a minor arc and a major arc. If neither is specified, it will be understood that the minor arc is the one referred to.

Introduction to Theorems 2 to 13

When a chord of a circle is drawn and the extremities are joined to the center, the chord, the arc, and the central angle are closely related to each other. One may say that the chord *corresponds* to the arc. With this understanding it will be shown that if in the same or equal circles two {chords / arcs / angles at the center} are equal, then the other two pairs of corresponding parts are also equal. For example, if two arcs are equal, the corresponding angles and corresponding chords are equal.

On the other hand, if two chords are unequal, it will be shown that the corresponding arcs and angles at the center are also unequal in the same order. The relation that exists between different parts of a figure, so that equality or inequality of two corresponding parts implies equality or inequality in other corresponding parts, is extremely useful.

EXERCISES

1. Construct two equal circles. With the aid of a protractor construct a central angle in each circle equal to 50°. Draw the chords of the minor arcs that are formed. Measure these chords and compare results.

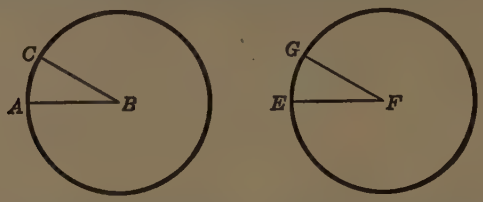
2. Construct two equal circles. In one circle construct a central angle of 40° and in the other construct a central angle of 55°. Draw the chords of the minor arcs formed, measure them, and compare results.

3. In a circle construct equal chords. Draw the central angles. Measure them and compare results.

4. Repeat Exercise 3 with unequal chords.

Theorem 2

181. *In the same circle or in equal circles, if two central angles are equal, their intercepted arcs are equal.*



Given the two equal ⓢ B and F, with equal central $\measuredangle$ B and F which have the arcs AC and EG respectively.

To prove that arc AC = arc EG.

Method. Superposition, using assumption 3, § 171.

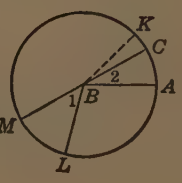
Proof

1. Place $\odot$ F on $\odot$ B so that the point F falls on the point B and $\angle F$ coincides with $\angle B$.	1. § 51, Postulate 7, and, by hypothesis, $\angle B = \angle F$.
2. Then E falls on A and G falls on C.	2. § 171, 1. Radii of equal ⓢ are equal.
3. No point of arc EG can fall outside or inside of arc CA.	3. § 171, 3.
4. $\therefore$ arc EG = arc AC.	4. They coincide.

If $\angle B$ and $\angle F$ are in the same circle, one angle may be rotated about the center until it coincides with the other.

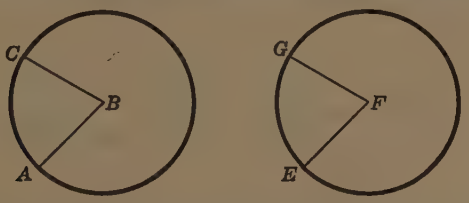
182. ***Corollary.*** *In the same circle or equal circles, if two central angles are unequal, the greater angle intercepts the greater arc.*

HINT. Let $\angle 1$ be greater than $\angle 2$. Lay off $\angle ABK$ equal to $\angle 1$. BK falls outside $\angle ABC$. Arc AK = arc ML. Etc.



Theorem 3 (*Converse of Theorem 2*)

183. *In the same circle or in equal circles, if two arcs are equal, they have equal central angles.*



Given the two equal ⑤ B and F, in which arc AC = arc EG.

To prove that $\angle F = \angle B$.

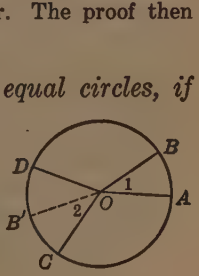
Method. Superposition, using assumption 3, § 171.

Proof

1. Place ⊙ F on ⊙ B so that FE falls on BA.	1. § 51, Postulate 7.
2. Then arc EG will fall on arc AC.	2. § 171, 3.
3. Point G will fall on C.	3. By hypothesis.
4. $\therefore$ GF coincides with CB.	4. § 51, Postulate 1.
5. $\therefore$ $\angle F = \angle B$.	5. They coincide.

If the arcs AC and EG are in the same circle, one may be rotated about the center until it coincides with the other. The proof then follows as above.

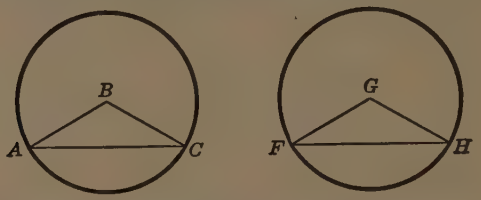
184. *Corollary. In the same circle or in equal circles, if two arcs are unequal, the greater arc has the greater central angle.*



HINT. Let arc AB be less than arc CD. Let arc CB' equal arc AB. Then point B' lies between C and D. Draw radius OB'. Then $\angle 1 = \angle 2$ and $\angle 2 < \angle COD$. Etc.

Theorem 4

185. *In the same circle or in equal circles, if two arcs are equal, their chords are equal.*



Given the two equal ⑤ B and G, in which arc AC = arc FH.

To prove that chord AC = ***chord*** FH.

Method. Prove that $\triangle ABC \cong \triangle FGH$.

Proof

1. Draw the radii AB, CB, FG, and HG.	1. § 51, Postulate 1.
2. Arc AC = arc FH.	2. By hypothesis.
3. In $\triangle ABC$ and $\triangle FGH$, $\angle B = \angle G$.	3. § 183. In the same ⊙ or equal ⑤, if two arcs are equal, they have equal central ∡.
4. $BA = GF = BC = GH$.	4. § 171, 1.
5. $\triangle ABC \cong \triangle FGH$.	5. § 62. s.a.s. = s.a.s.
6. $\therefore$ chord AC = chord FH.	6. § 64. Corresponding parts of congruent ▲ are equal.

If arcs AC and FH are in the same circle, the proof is identical with the above.

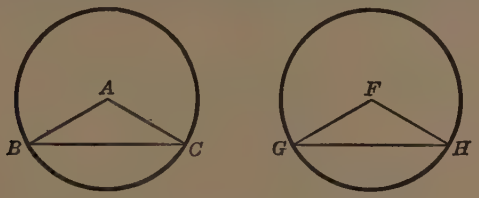
QUERY 1. Is there any chord of the circles for which the foregoing proof does not apply?

QUERY 2. What assumption applies to the case when the arcs in the foregoing theorem are semicircles?

QUERY 3. How would you prove this theorem for the major arcs AC and FH?

Theorem 5 (*Converse of Theorem 4*)

186. *In the same circle or in equal circles, if two chords are equal, their subtended arcs are equal.*



Given two equal ⑤ A and F, in which chord BC = chord GH.

To prove that arc BC = arc GH.

Method. Prove $\triangle ABC \cong \triangle FGH$; then $\angle A = \angle F$. It then follows that arc BC = arc GH.

Proof

1. Draw radii AB, AC, FG, and FH.	1. § 51, Postulate 1.
2. Chord BC = chord GH.	2. By hypothesis.
3. $AB = FG$ and $AC = FH$.	3. § 171, 1. Radii of equal ⑤ are equal.
4. $\triangle ABC \cong \triangle FGH$.	4. § 71. s.s.s. = s.s.s.
5. $\angle A = \angle F$.	5. § 64. Corresponding parts of congruent ▲ are equal.
6. $\therefore$ arc BC = arc GH.	6. § 181.

EXERCISES

1. Chords AB, BC, CD, DE, and EA of a circle are equal. Prove chord BD = chord BE.

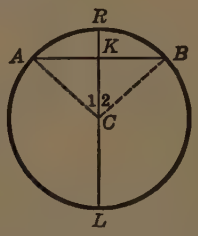
2. AC and RD are intersecting diameters. Prove $ABCD$ a rectangle.

3. In the adjacent figure, $AB = KR$ and $LB = LR$.

Prove $\triangle ARL \cong \triangle BKL$.

Theorem 6

187. *If a line passes through the center of a circle and is perpendicular to a chord, it bisects the chord and the arcs subtended by it.*



Given LR **passing through** C**, the center of the circle, and** $\perp$ **to chord** AB **at** K**.**

To prove that $AK = KB$***, arc*** AR $=$ ***arc*** BR***, and arc*** AL $=$ ***arc*** BL***.***

Method. Draw the radii AC and BC, and prove that $\triangle AKC \cong \triangle BKC$. Then consider $\angle 1$ and $\angle 2$ and their intercepted arcs.

Proof

1. Draw the radii AC and BC.	1. § 51, Postulate 1.
2. In the rt. $\triangle AKC$ and BKC, $CK = CK$.	2. Identical.
3. $CA = CB$.	3. § 171, 1.
4. $\triangle AKC \cong \triangle BKC$.	4. § 109.
5. $\therefore AK = KB$ and $\angle 1 = \angle 2$.	5. § 64.
6. Then arc AR $=$ arc RB.	6. § 181.
7. $\angle ACL = \angle BCL$.	7. § 39 and § 50, Axiom 4. Supplements of equal $\measuredangle$ are equal.
8. $\therefore$ arc AL $=$ arc BL.	8. § 181.

QUERY. Is the foregoing theorem true when the chord AB is a diameter?

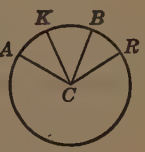
EXERCISES

1. Two intersecting diameters divide a circle into two pairs of equal arcs.

2. Two perpendicular diameters divide a circle into four equal arcs.

3. If three diameters, AB, FG, and KR, divide a circle into six equal arcs, prove that the six angles at the center contain 60° each.

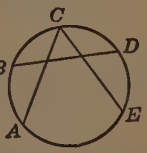
4. In the figure, A, K, B, and R are points in order on a circle with center C such that arc AK = arc BR.



Prove $\angle KCR = \angle ACB$.

5. AB is a chord of a circle equal to its radius.

Prove that the arc AB is one sixth of the whole circumference.



6. If the arcs AB, BC, CD, and DE of the same circle are equal, then the chords AC, BD, and CE are equal.

7. K, R, and L are three points on a circle such that arcs KR, RL, and LK are equal. The chords KR, RL, and LK are drawn.

Prove that $\triangle KRL$ is equiangular.

8. If a radius of a circle bisects a chord, it is perpendicular to the chord and bisects the minor arc.

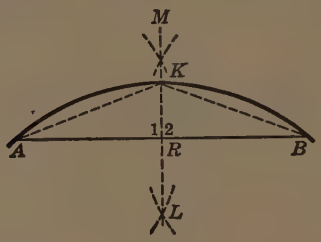
HINT. In a figure like that of § 187 prove the angles at K equal.

9. If a diameter bisects an arc, it bisects the chord of the arc at right angles.

10. If a line bisects a chord and one of its arcs, it is perpendicular to the chord, passes through the center, and bisects the other arc.

Construction 1

188. *Bisect a given circular arc.*



Given the arc AB.

Required to bisect AB.

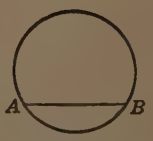
Method. Draw chord AB, and construct LM, its perpendicular bisector. Let LM cut the arc AB at K. Then K is the mid-point of arc AB.

Proof

1. In $\triangle ARK$ and BRK, $\angle 1 = \angle 2$.	1. By construction.
2. $KR = KR$.	2. Identical.
3. $AR = BR$.	3. By construction.
4. $\triangle ARK \cong \triangle BRK$.	4. § 62. s.a.s. = s.a.s.
5. $\therefore AK = BK$.	5. § 64.
6. $\therefore$ arc AK = arc BK.	6. § 186.

EXERCISES

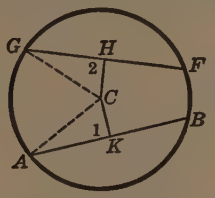
1. Given chord AB in a circle. Find two points on the circle each equidistant from A and B.



2. Given a circular arc ABC. How can the center of a circle of which it is a part be found? Prove.

Theorem 7

189. *In the same circle or in equal circles, if two chords are equal, they are equally distant from the center.*



Given the equal chords GF and AB in $\odot C$, and CH and CK their respective distances from the center.

To prove that $CH = CK$.

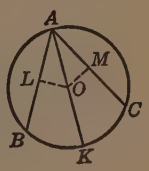
Method. Draw the radii CG and CA. Prove that $\triangle AKC \cong \triangle CHG$.

Proof

1. Draw radii CG and CA.	1. § 51, Postulate 1.
2. $\angle 1 = \angle 2 =$ 1 rt. $\angle$.	2. §§ 79 and 36.
3. Chord $AB =$ chord GF.	3. By hypothesis.
4. $GH = \frac{1}{2} GF$ and $AK = \frac{1}{2} AB$.	4. § 187.
5. $GH = AK$.	5. § 50, Axiom 6.
6. $CG = CA$.	6. § 171, 1.
7. $\triangle AKC \cong \triangle CHG$.	7. § 109.
8. $\therefore CH = CK$.	8. § 64.

EXERCISES

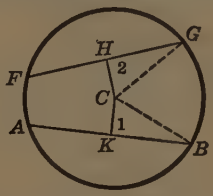
1. In the figure, AB and AC are equal and AK is a diameter. Prove $\angle BAK = \angle CAK$.



2. The arcs AB, BC, and CA of a circle are equal. Show that the distance of the chords AB, BC, and CA from the center is one half the radius.

Theorem 8 (*Converse of Theorem 7*)

190. *In the same circle or in equal circles, if two chords are equally distant from the center, they are equal.*



Given the chords *AB* and *FG* in ⊙ *C*, with their respective distances from the center, *CK* and *CH*, equal.

To prove that chord AB = chord FG.

Method. Draw the radii *CB* and *CG*. Prove that $\triangle BCK \cong \triangle GCH$. Then $BK = GH$ and $AB = FG$.

Proof

1. Draw *CB* and *CG*.	1. § 51, Postulate 1.
2. $CH = CK$.	2. By hypothesis.
3. $BC = GC$.	3. § 171, 1.
4. $\angle 1 = \angle 2 = 90°$.	4. §§ 79 and 36.
5. $\triangle BCK \cong \triangle GCH$.	5. § 109.
6. $\therefore BK = GH$.	6. § 64.
7. But $BK = \frac{1}{2} BA$ and also $GH = \frac{1}{2} GF$.	7. § 187.
8. $\therefore AB = FG$.	8. § 50, Axiom 5.

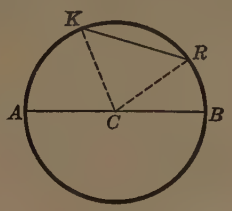
EXERCISES

1. Two chords from one extremity of a diameter making equal angles with it are equal.

2. If perpendiculars from the center to two chords are equal, the major arcs subtended by the chords are equal.

Theorem 9

191. *The diameter of a circle is greater than any other chord.*



Given the diameter AB and any chord KR.

To prove that $AB > KR$.

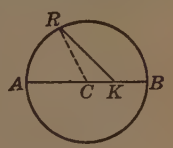
Method. Draw CK and CR, and show that their sum equals a diameter and is greater than KR.

Proof

1. Draw CK and CR.	1. § 51, Postulate 1.
2. $CK + CR > KR$.	2. § 163 (*b*).
3. $CK = CA = CR = CB$.	3. § 171, 1.
4. $\therefore CA + CB > KR$; that is, $AB > KR$.	4. Step 2 and § 50, Axiom 7.

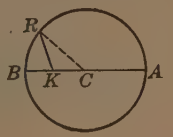
EXERCISES

1. The longest line to a circle from a point within it is the greater segment of the diameter passing through the point.



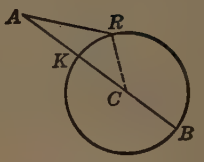
HINT. Let KR be any distance to the circle other than KA. Prove $KR < KA$.

2. The shortest line to a circle from a point within it is the shorter segment of the diameter passing through the point.



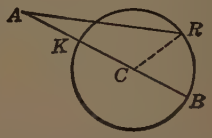
HINT. $RC - KC < KR$ and $BC - KC = BK$. Etc.

3. The shortest line to a circle from a point outside is the external segment of the line from the point passing through the center of the circle.



HINT. Draw CR and prove $AR > AK$.

4. The longest line to a circle from a point outside is the line from the point passing through the center of the circle.

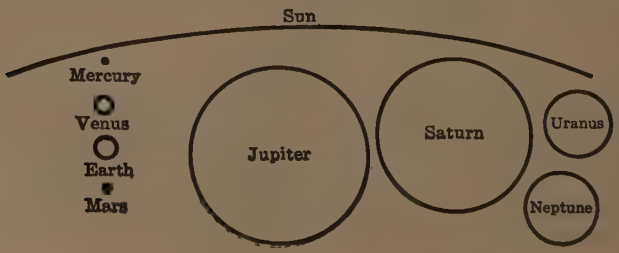


HINT. Draw CR and prove $AB > AR$.

5. Given a circle and chord AB. Equal chords AD and BC are drawn from A and B respectively. Then DC is drawn.

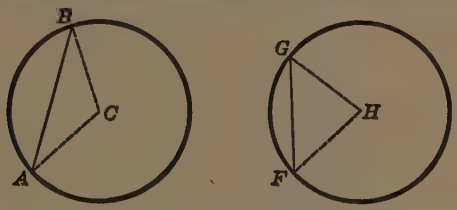
Prove that the figure $ABCD$ is a trapezoid or a rectangle.

192. Comparative size of the members of the sun's family. The sun, the earth, the moon, and the planets are nearly spherical in shape. When one looks at a sphere the outline appears to be a circle. The relative apparent size of the planets can therefore be represented as in the figure below by circles having the proper radius. The sun is so large compared with the smaller planets that the complete circle representing it cannot be shown. Illustrations like these convey a more vivid notion of the relative size of such large objects than do their diameters given in miles.



Theorem 10

193. *In the same circle or in equal circles, if two minor arcs are unequal, the greater arc has the greater chord.*



Given two equal ⑤ C and H, in which minor arc $AB >$ minor arc FG, and chords AB and FG.

To prove that chord $AB >$ chord FG.

Method. In $\triangle ABC$ and FGH prove that $AC = FH$, $BC = GH$, and $\angle C > \angle H$. The conclusion then follows.

Proof

1. Draw radii AC, BC, FH, and GH.	1. § 51, Postulate 1.
2. In $\triangle ABC$ and $\triangle FGH$, $AC = FH$ and $BC = GH$.	2. § 171, 1.
3. $\angle C > \angle H$.	3. § 184.
4. $\therefore$ chord $AB >$ chord FG.	4. § 167.

QUERY. If the two unequal arcs were in the same circle, would the preceding proof hold without any change?

EXERCISES

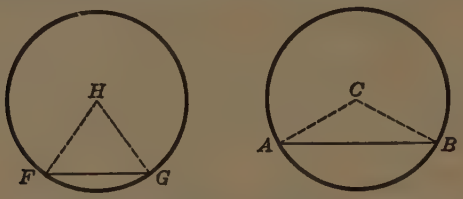
1. AK is a diameter, and KB and KC are chords such that KB lies between AK and KC.

Prove chord $KB >$ chord KC.

2. Change the word "minor" to "major" in Theorem 10 and make any other changes needed in order to have the resulting statement true.

Theorem 11 (*Converse of Theorem 10*)

194. *In the same circle or in equal circles, if two chords are unequal, the greater chord has the greater minor arc.*



Given two equal ⊙ C and H, in which chord $AB >$ chord FG.

To prove that arc $AB >$ arc FG.

Method. In △ ABC and FGH, show that $AC = FH$, $BC = GH$, and $AB > FG$. Then prove that $\angle C > \angle H$ and hence arc $AB >$ arc FG.

Proof

1. Draw radii AC, BC, FH, and GH.	1. § 51, Postulate 1.
2. In △ ABC and △ FGH, $AC = FH$ and $BC = GH$.	2. § 171, 1.
3. Chord $AB >$ chord FG.	3. By hypothesis.
4. $\angle C > \angle H$.	4. § 168.
5. ∴ arc $AB >$ arc FG.	5. § 182.

EXERCISES

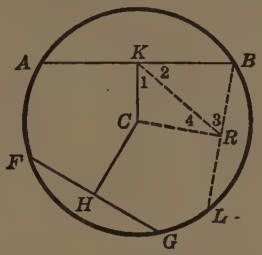
1. If AB is a diameter, and BR and BC are chords with BC the greater, prove that chord AC is less than chord AR.

2. If the diameters AC and BD are not perpendicular to each other, prove that the adjacent sides of the quadrilateral $ABCD$ are not equal.

3. Construct an arc of a circle equal to a given arc.

Theorem 12

195. *In the same circle or in equal circles, if two chords are unequal, the greater is at the less distance from the center.*



Given in $\odot C$ chord $AB >$ chord FG, and their respective distances from the center, CK and CH.

To prove that $CK < CH$.

Method. Draw chord BL equal to chord FG, and $CR \perp$ to BL. Join K and R. Show that $\angle 3 > \angle 2$ in $\triangle BKR$. Then show that $\angle 1 > \angle 4$ and that $CR > CK$, and hence $CK < CH$.

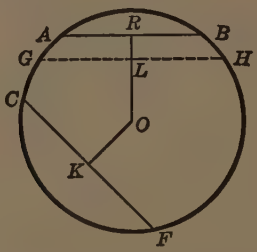
Proof

1. Let BL be a chord equal to chord FG, and let CR be $\perp$ to BL. Join K and R.	1. § 76 and § 51, Postulate 1.
2. $KB = \frac{1}{2} AB$ and $BR = \frac{1}{2} BL$.	2. § 187.
3. Chord $AB >$ chord FG or its equal, chord BL.	3. By hypothesis.
4. $KB > BR$.	4. § 162, Axiom 11.
5. $\angle 3 > \angle 2$.	5. § 164.
6. $\angle 3 + \angle 4 = \angle 2 + \angle 1$.	6. § 36.
7. $\angle 4 < \angle 1$.	7. § 162, Axiom 13.
8. $\therefore CK < CR$.	8. § 165.
9. But $CR = CH$.	9. § 189.
10. $\therefore CK < CH$.	10. Steps 8 and 9.

QUERY. Does Theorem 12 hold when one chord is a diameter?

Theorem 13

196. *In the same circle or in equal circles, if two chords are unequally distant from the center, the chord at the greater distance is the less.*



Given chords AB and CF, with distance $OR >$ distance OK.

To prove that *chord* $AB <$ *chord* CF.

Method. Lay off $OL = OK$, and draw through L chord $GH \perp$ to OR. Then prove that chord $CF =$ chord GH and chord $GH >$ chord AB.

Proof

1. On OR lay off $OL = OK$ and draw chord $GH \perp$ to OR through L.	1. § 51, Postulate 6.
2. Then $GH = CF$.	2. § 190.
3. AB is $\parallel$ to GH.	3. §§ 79 and 83.
4. Since $OK = OL < OR$, L lies between O and R.	4. By construction and hypothesis.
5. $\therefore$ arc $GH >$ arc AB.	5. § 162, Axiom 8.
6. $\therefore$ chord $GH >$ chord AB.	6. § 193.
7. $\therefore$ chord $CF >$ chord AB.	7. § 50, Axiom 7.

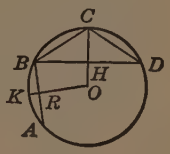
QUERY. How would the foregoing proof be changed if the chords AB and CF were in equal circles?

EXERCISES ON CHORDS, ARCS, AND CENTRAL ANGLES

1. Does Theorem 13 hold when one chord passes through the center? Prove.

2. ABC is a triangle whose sides are chords of a circle with AB farther from the center than AC. Which angle is the greater, B or C? Prove.

3. Arcs AB, BC, and CD are equal. K is the mid-point of arc AB, KR is $\perp$ to chord AB at R, and CH is $\perp$ to chord BD at H. Prove $CH > KR$.



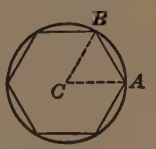
4. AB is a diameter, O is the center, and BK and BR are chords of a circle. LO is $\perp$ to BK at L and OH is $\perp$ to BR at H, and $OL < OH$.

Prove chord $AK <$ chord AR.

5. Two intersecting chords making equal angles with the diameter passing through their point of intersection are equal. Prove.

6. If a diameter is drawn through a point (not the center) within a circle equidistant from two points upon it, the arc between the two points is bisected. Prove.

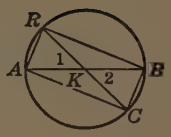
7. If a circle is divided into six equal arcs, any chord joining adjacent points of division is equal to the radius of the circle. Prove.



8. If a line bisects an arc and is perpendicular to the chord of the arc, it will, if produced, pass through the center of the circle. Prove.

9. Two points, each equidistant from the ends of a chord, determine a line passing through the center of the circle.

10. If two chords of a circle bisect each other, they are diameters. Prove.



HINT. Prove $\triangle AKR \cong \triangle BKC$ and arc AR = arc BC. Then prove arc BR = arc AC in like manner.

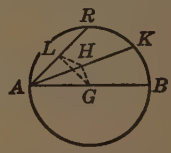
11. A is the center of a circle and K is a point outside. KRL and KHG make equal angles with AK and cut the circle in R and L and in H and G respectively.

Prove chord RL = chord HG.

12. B is the mid-point of the arc AC of a circle, and BK and BR are drawn perpendicular to the radii OA and OC respectively.

Prove $BK = BR$.

13. AB is a diameter, G the center of a circle, and AK and AR chords such that K lies between B and R. The line GH is $\perp$ to AK at H, and GL is $\perp$ to AR at L. The line HL is drawn.



Prove $\angle LHG > \angle GLH$.

14. If, from the extremities of any diameter, perpendiculars are drawn upon any chord (produced if necessary), the feet of the perpendiculars are equidistant from the center of the circle. Prove.

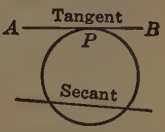
15. If two unequal chords intersect in a circle, the greater chord makes the less acute angle with the diameter through the point of intersection. Prove.

16. A and B are the centers of two circles which intersect in K and R. KL is drawn to the mid-point of AB. A line through $K \perp$ to KL cuts one circle in H and the other in G.

Prove chord KH = chord KG.

HINT. Use §§ 144, 187.

197. Tangent. A *tangent* to a circle is a straight line which lies in the plane of the circle and, however far produced, has only one point in common with the circle.



If a straight line is tangent to a circle, the circle is also tangent to the line.

The point common to the circle and to the tangent is called the point of tangency or point of contact.

Thus, in the accompanying figure, AB is the tangent and P the point of contact.

198. Secant. A *secant* is a line which cuts a circle in two points.

Circumscribed polygon

199. Circumscribed polygon. A polygon whose sides are tangent to a circle is called a *circumscribed polygon.*

Inscribed polygon

200. Inscribed polygon. A polygon whose vertices lie on a circle is called an *inscribed polygon.*

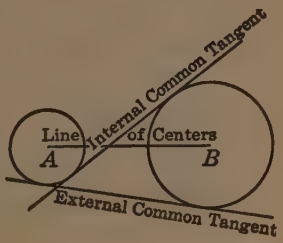
Concentric circles

201. Concentric circles. Two or more circles which have the same center are called *concentric circles.*

202. Common tangent. A *common tangent* to two circles is a straight line tangent to each.

203. Internal common tangent. A common tangent of two circles which passes between them is an *internal common tangent.*

204. External common tangent. A common tangent which does not pass between two circles is an *external common tangent.*



205. Line of centers. The line segment joining the centers of two circles is called the *line of centers.*

206. Tangent circles. If two circles have only one point in common, each is *tangent* to the other.

Externally tangent circles

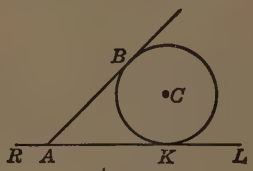
207. Externally tangent circles. If each of two tangent circles is outside the other, they are tangent *externally*.

208. Internally tangent circles. If one of two tangent circles is within the other, they are tangent *internally*.

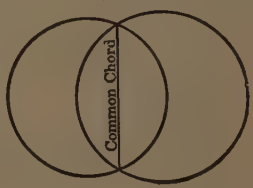
Internally tangent circles

209. Tangent from an outside point. A tangent to a circle from an outside point is a line segment whose extremities are the outside point and the point of contact.

In the adjacent figure the line segments AB and AK are tangents to the circle from the outside point A.



210. Common chord. The *common chord* of two intersecting circles is the chord joining the points of intersection.



QUERY 1. Can two circles have no common tangent? If so, when?

QUERY 2. Can two circles have only one common tangent? If so, when?

QUERY 3. Can two circles have only two common tangents? If so, when?

QUERY 4. Can two circles have only three common tangents? If so, when?

QUERY 5. Can two circles have only four common tangents? If so, when?

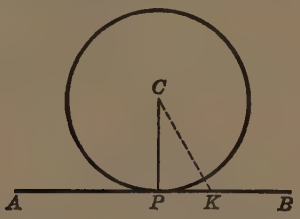
QUERY 6. Can two circles have more than four common tangents? If so, when?

QUERY 7. Can three circles be so situated as to have only one tangent common to all three?

QUERY 8. Can three circles be so placed as to have no tangent common to any two of them?

Theorem 14

211. *If a line is perpendicular to a radius at its outer extremity, it is tangent to the circle.*



Given CP, the radius of $\odot C$, and $AB \perp$ to CP at P.

To prove that AB ***is tangent to the circle.***

Method. Draw CK, any line other than the radius CP, to AB, and prove that K is outside the circle.

Proof

1. Draw CK to K, any point on AB except P.	1. § 51, Postulate 1.
2. $CK > CP$.	2. § 166.
3. K is outside the $\odot$.	3. § 171, 5.
4. Hence any point of AB except P is outside the $\odot$, and AB is tangent to the $\odot$.	4. § 171, 5, and § 197.

EXERCISES

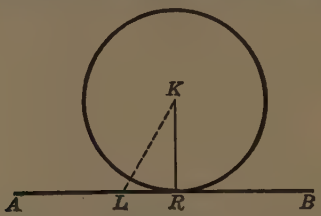
1. Construct a tangent to a circle at a point on the circle.

2. Perpendiculars to a diameter at its extremities are parallel tangents. Prove.

3. Points A, B, C, D, and E divide a circle into five equal parts. Lines perpendicular to the radii at these points form a circumscribed polygon. Is this polygon equiangular? Prove.

Theorem 15 (*Converse of Theorem 14*)

212. *If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of tangency.*



Given $\odot K$, **the line** AB **tangent to it at** R, **and radius** KR.

To prove that KR ***is*** $\perp$ ***to*** AB.

Method. Draw KL to L, any point on AB except R, and prove that $KL > KR$.

Proof

1. Draw KL to L, any point on AB except R.	1. § 51, Postulate 1.
2. $KL > KR$.	2. § 171, 5.
3. Hence KR is the shortest line from K to AB, and KR is $\perp$ to AB.	3. § 166.

EXERCISES

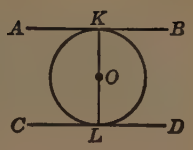
1. Tangents to a circle at the extremities of a diameter are parallel to each other. Prove.

2. $ABCDEFGH$ is a circumscribed equiangular octagon. OK and OR are radii to the points of contact of sides AB and BC. How many degrees are there in $\angle KOR$?

3. Tangents to a circle at the extremities of two perpendicular diameters meet in G, H, L, and M.

Prove that $GHLM$ is a circumscribed square.

4. If two tangents to a circle are parallel, the points of contact and the center are in a straight line.



HINT. Prove that KO extended is $\perp$ to CD and that it must coincide with OL.

5. Prove that the quadrilateral formed by joining the points of contact of a square circumscribed about a circle is a square.

CONSTRUCTIONS

1. Given chord AB of a circle, construct another chord of the circle equal to it.

2. Construct in a circle a chord equal to the sum of two given chords. Is this construction always possible?

3. Construct a chord of a circle half as far from the center as a given chord.

4. Given arc AB in a circle. Lay off another arc in the same circle equal to half of it.

5. Construct in a circle an arc equal to twice a given arc. Is this construction always possible?

6. Construct two parallel tangents to a given circle.

7. Construct two tangents to a circle which are perpendicular to each other.

8. Construct a square circumscribed about a given circle.

HINT. First construct two perpendicular diameters of the circle.

9. Construct a rhombus circumscribed about a given circle.

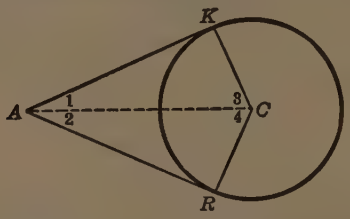
HINT. First draw two diameters which are not perpendicular to each other.

10. Inscribe a square in a circle.

11. Construct two tangents to a circle, making an angle of 60° with each other.

Theorem 16

213. *If two tangents are drawn to a circle from an outside point, they are equal.*



Given AK **and** AR**, tangents from the point** A **to** $\odot C$.

To prove that $AK = AR$.

Method. Draw CK, CR, and CA. Then prove that $\triangle AKC \cong \triangle ARC$.

Proof

1. Draw radii CK and CR and line AC.	1. § 51, Postulate 1.
2. In $\triangle AKC$ and $\triangle ARC$, $\angle K = \angle R$.	2. §§ 212 and 36.
3. $CA = CA$.	3. Identical.
4. $CK = CR$.	4. § 171, 1.
5. $\triangle AKC \cong \triangle ARC$.	5. § 109.
6. $\therefore AK = AR$.	6. § 64.

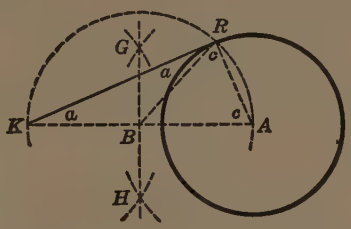
214. ***Corollary.*** *The line joining the point of intersection of two tangents to a circle with the center bisects the angle between the tangents and the angle between the radii drawn to the points of contact.*

QUERY 1. What relation does the line drawn from the center of the circle to the point of intersection of two tangents to the circle bear to the chord joining the points of tangency?

QUERY 2. In the figure above, how is the arc KR divided by the line AC?

Construction 2

215. *Construct a tangent to a circle from an outside point.*



Given a circle with center A, and a point K outside it.

Required to construct a tangent to $\odot A$ from K.

Method. Draw AK. Construct HG, the mid-perpendicular of AK. Let HG cut AK in B. With B as the center and BA as the radius, describe a semicircle, cutting the circle A at R. Draw KR. Then KR is the required tangent.

Proof

1. Draw AR and BR.	1. § 51, Postulate 1.
2. Then $BA = BR = BK$.	2. § 171, 1.
3. $\therefore \angle a = \angle a$ and $\angle c = \angle c$.	3. § 67.
4. In $\triangle AKR$, $2\angle a + 2\angle c = 180°$, or $\angle a + \angle c = \angle KRA = 90°$.	4. § 100 and § 50, Axiom 6.
5. $\therefore KR$ is $\perp$ to AR and is tangent to the $\odot$.	5. § 211.

QUERY 1. How could the construction given above be extended so as to obtain the other tangent from K to $\odot A$?

QUERY 2. If one had a number of concentric circles, how could one construct tangents to each from a point outside all of them by the method given above?

QUERY 3. If, in the construction above, the point B happens to fall on the circle, how many degrees are there in $\angle K$?

EXERCISES

1. The line AB touches a circle at K, the line AL at L, and the line BR at R.

Prove $AB = AL + BR$.

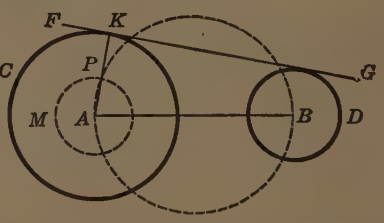
2. In a circumscribed quadrilateral the sum of two opposite sides equals the sum of the other two.

3. Two circles have two common tangents which do not pass between the circles. The points of tangency are A and B for one tangent and C and D for the other.

Prove $AB = CD$.

4. Given A and B any two points outside $\odot C$. Circumscribe about $\odot C$ a quadrilateral two of whose vertices are points A and B.

5. In the adjacent figure, AC and BD are circles whose radii are R and r respectively. Circle AM is drawn with center A and radius $R - r$. A fourth circle on AB as a diameter cuts $\odot AM$ at P. Through P the radius AK of $\odot AC$ is drawn, and FG is drawn through K perpendicular to AK.



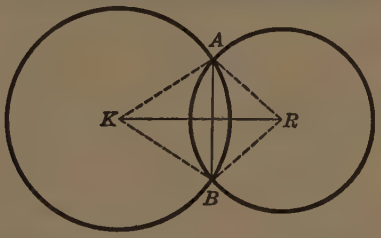
Prove that FG is tangent to the two ⓢ AC and BD.

HINT. Draw BP and a radius through B parallel to AK.

6. Two circles are tangent at A. If any point B on their common internal tangent is taken as a center and BA as a radius and an arc is drawn cutting one circle in K and the other in L, BK is tangent to one circle and BL to the other. Prove.

Theorem 17

216. *If two circles intersect, the line of centers bisects their common chord at right angles.*



Given circles intersecting at A and B, the line of centers KR, and the common chord AB.

To prove that KR bisects AB at right angles.

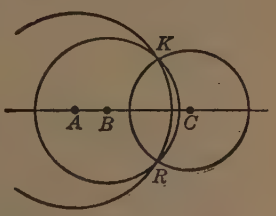
Method. Prove that K and R are equidistant from the ends of the common chord and therefore KR is the mid-perpendicular of AB.

Proof

1. Draw the radii KA, KB, RA, and RB.	1. § 51, Postulate 1.
2. Then $KA = KB$ and $RA = RB$.	2. § 171, 1.
3. Therefore KR is the mid-perpendicular of AB.	3. § 77.

EXERCISES

1. In the figure prove centers A, B, and C are in a straight line.

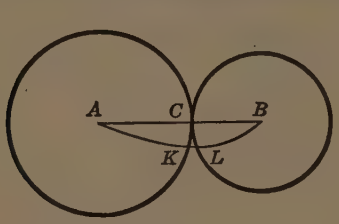


2. A is the center of a circle. The radius AB is produced to C. With C as the center and CB as the radius another circle is described

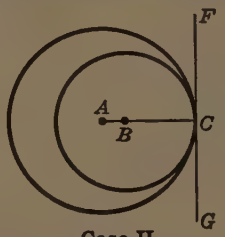
Prove that the two circles are tangent.

Theorem 18

217. *If two circles are tangent externally or internally, the line of centers passes through the point of tangency.*



Case I



Case II

Case I. Given ⓢ A and B tangent externally at C.

To prove that ACB is a straight line.

Method. Draw any other line from A to B, and prove that it is longer than ABC.

Proof

1. Draw the radii AC and BC.	1. § 51, Postulate 1.
2. Draw any other line from A to B which does not go through C. Let this line cut ⊙ A in K and ⊙ B in L. Then AK is equal to or is longer than AC, and BL is equal to or is longer than BC.	2. The shortest line from A to K is equal to the radius of ⊙ A and from B to L is equal to the radius of ⊙ B.
3. Now $KL > 0$.	3. KL is outside the ⓢ, since they are tangent only at C.
4. $\therefore AK+KL+LB > AC+BC$; that is, $AKLB > ACB$.	4. § 162, Axioms 10 and 12.
5. $\therefore$ ACB is the shortest line from A to B and hence is straight.	5. § 163 (d).

Case II. Given ⓢ A and B tangent internally at C.

To prove that ABC is a straight line.

Method. Draw a tangent at C to $\odot A$, and prove it tangent to $\odot B$. Then prove that radii AC and BC coincide.

Proof

1. Draw FG tangent to $\odot A$ at C and radii AC and BC.	1. § 215, and § 51, Postulate 1.
2. Then FG has only one point in common with $\odot B$ and hence is tangent to it.	2. § 197. For $\odot B$ is entirely inside $\odot A$.
3. $\therefore AC$ and BC are $\perp$ to FG at C and coincide.	3. § 212 and § 51, Postulate 6.
4. $\therefore ABC$ is a straight line.	4. Step 3.

218. *Corollary 1. The line of centers of two externally tangent circles is equal to the sum of the two radii.*

219. *Corollary 2. The line of centers of two internally tangent circles is equal to the difference of the two radii.*

220. *Corollary 3. A line through the point of contact of two tangent circles perpendicular to their line of centers is a common tangent of the two circles.*

EXERCISES

1. A is the center of a circle. K is a point on the radius AB. With K as the center and KB as the radius, another circle is described.

Prove that the two circles are tangent.

2. Given two circles ($R > r$) tangent internally.

Prove that the largest circle tangent to the greater circle internally and to the smaller circle externally has a radius $R - r$.

CONSTRUCTIONS

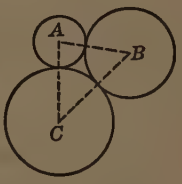
1. Given two points as centers, draw two circles tangent externally.

2. Given two points as centers, draw two circles tangent internally.

3. Given the two radii, draw two circles tangent externally.

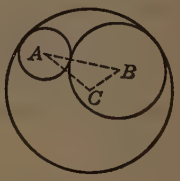
4. Given two radii, draw two circles tangent internally, one with its center at a given point on a given line and the other with center on the given line.

5. Given three radii, draw three circles tangent externally.



HINT. Suppose the circles drawn. How can $\triangle ABC$ be constructed from the given data? How long is AB? BC? CA? Can $\triangle ABC$ always be constructed? Explain.

6. Given the three radii, construct a circle tangent to and containing two given tangent circles.



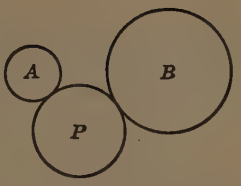
HINT. Suppose the circles constructed. In $\triangle ABC$ how long is AB? BC? CA? Can $\triangle ABC$ always be constructed from the given data? Explain.

7. Given two intersecting circles. Construct a third circle tangent internally to the first two.

8. Given two tangent circles. Construct a circle of given radius tangent internally to one and externally to the other. Is this problem always possible? How many solutions are possible?

9. Given two equal circles not tangent. Construct a circle tangent externally to the two. How many such circles are possible?

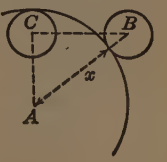
10. Given two unequal circles which are not tangent. Construct a circle tangent externally to the two.



HINT. How may a point P be located which is the same distance from $\odot A$ and $\odot B$? How many solutions are possible?

11. Given two fixed unequal circles. Construct a circle of given radius tangent externally to the two. Is the problem always possible? How many circles are possible?

12. Given two equal circles not tangent. Construct a circle tangent internally to one and externally to the other.



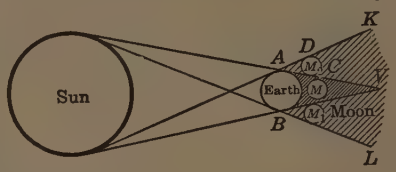
HINT. Suppose the circle constructed. How long is AB? BC? CA? How can $\triangle ABC$ be constructed from the given data? How many such circles are possible?

221. The earth's shadow cone. The adjacent figure indicates the shape of the earth's shadow. One's shadow on the ground or that of a house or a tree seems flat. The astronomer, however, is interested in solid shadows — shadows having three dimensions. When he thinks of the shadow of the earth he thinks of the portion of space from which the light of the sun is cut off. Since light travels in straight lines this is the cone ABC.

222. Eclipse of the moon. The moon's path round the earth is nearly circular. The radius of the circle is about 240,000 miles. While moving thus, the two travel round the sun, at a distance of about 93,000,000 miles. Occasionally the centers of the earth, the moon, and the sun are in, or very nearly in, a straight line. At such times the earth may be between the sun and the moon, which

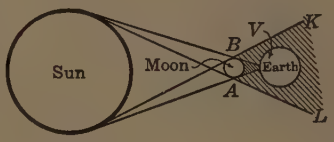
causes an eclipse of the moon. Or the moon may be between the earth and the sun, causing an eclipse of the sun.

The adjacent figure shows the moon at M wholly eclipsed, since no light from the sun enters the region AVB. Within the region KAV the sun is only partly visible. But the amount of the sun visible increases from C toward D. The same is true of the region VBL.



Hence when the moon is at M_1 or M_2 it is partially eclipsed. Before it crosses the line BL and after it crosses the line AK it is not in any shadow. Thus the internal and external common tangents to the two circles play an important part in understanding an eclipse of the moon.

223. Eclipse of the sun. Here the moon cuts off the light from some portion of the earth's surface. Since the moon is much smaller than the earth, the path of the shadow on the earth is only a few miles wide. This also accounts for the fact that in a given locality an eclipse of the sun is less frequent than one of the moon.

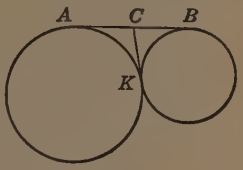


Where the shadow cone AVB touches the earth it traces a path where, for a short time, the sun is wholly invisible — "the path of totality," as the astronomer calls it. Outside this path a portion of the sun can be seen; that is, the sun is partially eclipsed. Here again, as in the case of the eclipse of the moon, the position of the internal or the external common tangents to the two circles determines the regions on the surface of the earth in which the eclipse is partial or total.

EXERCISES ON TANGENTS AND TANGENT CIRCLES

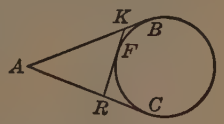
1. If two circles are tangent externally, two tangents drawn to them from any point in their common internal tangent are equal. Prove.

2. If two circles are tangent externally, their common internal tangent bisects the portion of their common external tangent between the points of contact. Prove.



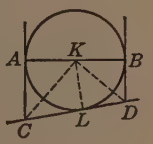
3. AB and AC are tangent to a circle at B and C respectively. Prove that AK, a line from A to the center, is the mid-perpendicular of BC.

4. AB and AC are two tangents to a circle at B and C respectively. A third line tangent to the arc BC at F cuts AB at K and AC at R. Prove that the perimeter of $\triangle AKR$ is equal to AB plus AC.



5. The segments of two internal common tangents of two circles included between their respective points of contact are equal. Prove.

6. Tangents CA and DB of a circle drawn at the extremities of the diameter AB meet a third tangent CD at C and D respectively. K is the center of the circle. Prove that $\angle CKD$ is a right angle.



HINT. Draw KL to the point of contact of CD.

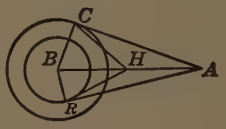
7. Tangents to a circle at the extremities of any pair of diameters which are not perpendicular to each other form a rhombus. Prove.

8. If the opposite sides of a circumscribed quadrilateral are parallel, the four sides are equal. Prove.

9. The contact points of the exterior common tangents of two circles are A and B for one circle and C and D for the other.

Prove AB is $\parallel$ to CD.

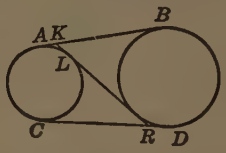
10. Tangents from the same point to two concentric circles cannot be equal.



HINT. Let AC and AR be the tangents, B the center, and H the mid-point of AB. Draw HC and HR. Then study $\triangle HBC$, HBR, AHC, and AHR.

11. If a point lies on each of two circles and also on the line of centers, the circles are tangent to each other. Prove.

12. A and B are the points of contact of a common exterior tangent to two circles. One common interior tangent cuts AB at K and the other common exterior tangent at R.



Prove $AB = KR$.

MEASUREMENT OF ANGLES BY ARCS OF CIRCLES

224. Degree of arc. If an arc whose central angle is a right angle is divided into ninety equal parts, one of these parts is called a *degree of arc.*

From the preceding definition and § 37 it follows that a central angle of one degree intercepts on the circle an arc of one degree.

Since two perpendicular diameters form four central angles each of which is a right angle, it follows that in any circle there are three hundred and sixty degrees of arc.

225. Relation of a central angle to its intercepted arc. The number of angle degrees in a central angle is equal to the number of arc degrees in its intercepted arc.

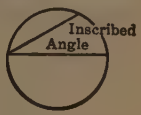
The usual statement of this relation is the theorem: *A central angle is measured by its intercepted arc.*

The truth of this theorem will be assumed.

The same number, therefore, expresses the measure of an arc of a circle and of its subtended central angle, but this does not say that an angle is the same thing as an arc. It merely asserts that they may be equal numerically, just as the number of acres in a field may be equal numerically to the number of rods of fence around the field, without implying that the field and the fence or an acre and a rod are the same thing.

The length of an arc in feet or inches must not be confused with the number of degrees it contains. In the adjacent figure, the angle O is a central angle in both circles. Therefore arc AB and arc KR contain the same number of degrees. But arc AB is shorter than arc KR. In the same circle or in equal circles, however, two arcs of the same number of degrees are of equal length. Thus, if each of arcs AB and CD of the adjacent figure contains m degrees, they are also of equal length in feet or inches.

226. Inscribed angle. An angle whose vertex is on a circle and whose sides are chords of the circle is called an *inscribed angle.*



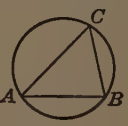
THE USE OF SIXTY IN CIRCLE DIVISION

The division of a right angle into ninety degrees is equivalent to dividing a circle into three hundred and sixty degrees. This part of our mathematical heritage, as well as the present division of the day into hours, minutes, and seconds, comes from the Babylonians. Why they adopted such a scheme of division we do not know. Cantor, a great historian of mathematics, suggests that their division of a circle came from their combining an astronomical belief with a geometrical fact. At first, according to his supposition, they believed that the sun went round the earth in a circle once in three hundred and sixty days, thus making its daily motion $\frac{1}{360}$ of the whole. Moreover, they were familiar with the fact that a chord of a circle equal to the radius has an arc which is exactly one sixth of the circle (page 169, Exercise 7). Such an arc seemed a natural division of the circle, since the sun would take sixty days to pass over it. In some such way they were led to what is called the sexagesimal division of the circle.

About the time the metric system was devised (1790), French mathematicians suggested that the division of a right angle be put on the decimal, or rather the centesimal, basis. The right angle was to be divided into 100 *grades*, each grade into 100 minutes, and each minute into 100 seconds. This method possessed many advantages over the established one. To make it effective, however, all mathematical tables, records of observations, and books of reference would have to be changed to the centesimal basis. The enormous labor involved makes it probable that this will never be done.

227. Segment. A segment of a circle is a figure formed by an arc of the circle and its chord.

An angle is said to be inscribed in a segment or an arc if its vertex is on the arc and its sides pass through the ends of the arc.



Thus, $\angle C$ may be spoken of as inscribed in the segment ACB or in the arc ACB.

228. Sector. A sector of a circle is a figure formed by two radii and their intercepted arc.

QUERY 1. How many degrees of angle are there about the center of a circle?

QUERY 2. How many degrees of arc are there in a circle?

QUERY 3. What figure may be called either a segment or a sector?

QUERY 4. A chord forms how many segments?

QUERY 5. Two radii form how many sectors?

EXERCISES

1. The circumference of a circle is 24 ft. How many degrees in an arc 6 ft. long? in one 8 ft. long?

2. An arc of a circle containing 50° is 12 ft. long; find the circumference of the circle.

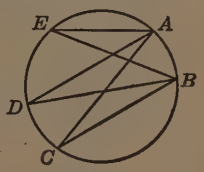
3. One side of an equilateral triangle inscribed in a circle has an arc 16 ft. long. Find the circumference of the circle.

Introduction to Theorems 19 and 20

EXERCISES

In each of the following exercises use a circle the same size as the protractor circle. Such a circle can be drawn with the compasses if the radius used equals the radius of the protractor circle.

1. Draw a circle, lay off an arc, AB, of 52°, and draw chords forming the inscribed angles C, D, and E. Measure each of the three angles with the protractor. Compare the number of degrees in each with the number in arc AB.

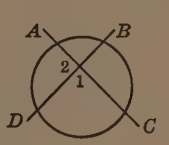


2. Repeat Exercise 1, making arc AB equal to 76°.

3. Complete the following statement so that it will agree with the results of Exercises 1 and 2:

The number of degrees in an inscribed angle is equal to ____ the number of degrees in its intercepted arc.

4. Draw a circle, lay off arc $AB = 46°$ and $CD = 88°$, and draw AC and BD. Measure $\angle 1$ and arcs AB and CD. What relation exists between the number of degrees in $\angle 1$ and the number in arcs AB and CD? Measure $\angle 2$ and arcs AD and BC, and compare the number of degrees in the arcs and the angles.



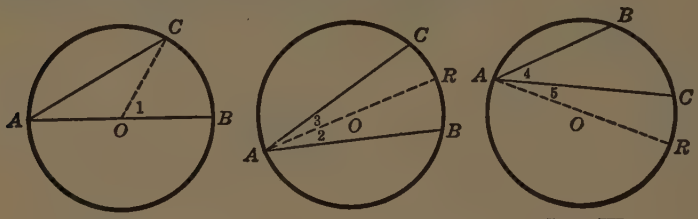
5. Repeat Exercise 4 with a different figure.

6. Complete the following statement in accord with the results of Exercises 4 and 5:

The number of degrees in each of two vertical angles formed by two intersecting chords is equal to ____ the number of degrees in the ____ of the intercepted arcs.

Theorem 19

229. *An inscribed angle is measured by one half the number of degrees in its intercepted arc.*



Case I Case II Case III

Given $\angle BAC$, **an inscribed angle.**

To prove that $\angle BAC$ ***is measured by*** $\frac{1}{2}$ ***arc*** BC.

Method. Case I, when one side of the angle is a diameter, is fundamental for Cases II and III.

In Case I draw the radius CO, and then show that $\angle A = \angle C = \frac{1}{2} \angle 1 = \frac{1}{2}$ arc BC. When, as in Case II, the center is inside the angle, or when, as in Case III, the center is outside the angle, draw the diameter and apply the result of Case I.

Proof

Case I. *When one side of the angle is a diameter*

1. Draw the radius CO.	1. § 51, Postulate 1.
2. Then $\triangle ACO$ is isosceles and $\angle A = \angle C$.	2. § 171, 1, and § 67.
3. Also, $\angle 1 = \angle A + \angle C$.	3. § 102.
4. Hence $\angle 1 = 2 \angle A$; that is, $\angle A = \frac{1}{2} \angle 1$.	4. § 50, Axiom 7.
5. $\angle 1$ is measured by arc BC.	5. § 225.
6. $\therefore \angle A$ is measured by $\frac{1}{2}$ arc BC.	6. Steps 4 and 5.

Case II. When the center is inside the angle

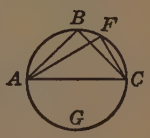
1. Draw diameter AR.	1. § 51, Postulate 1.
2. Then $\angle 2$ is measured by $\frac{1}{2}$ arc BR.	2. Case I.
3. Also, $\angle 3$ is measured by $\frac{1}{2}$ arc CR.	3. Case I.
4. $\therefore \angle 2 + \angle 3 = \angle BAC$ and is measured by $\frac{1}{2}$ arc BC.	4. Steps 2 and 3.

Case III. When the center is outside the angle

1. Draw diameter AR.	1. § 51, Postulate 1.
2. Then $\angle BAR$ is measured by $\frac{1}{2}$ arc BR.	2. Case I.
3. Also, $\angle 5$ is measured by $\frac{1}{2}$ arc CR.	3. Case I.
4. $\therefore \angle BAR - \angle 5 = \angle 4$ and is measured by $\frac{1}{2}$(arc BR − arc CR) $= \frac{1}{2}$ arc BC.	4. Steps 2 and 3.

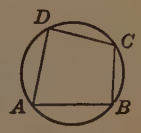
230. ***Corollary 1.*** *An angle inscribed in a semicircle is a right angle.*

231. ***Corollary 2.*** *Two angles inscribed in the same segment are equal.*



HINT. How is $\angle B$ measured? How is $\angle F$ measured? Compare results.

232. ***Corollary 3.*** *The opposite angles of an inscribed quadrilateral are supplementary.*

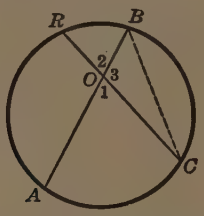


HINT. If $ABCD$ is an inscribed quadrilateral, what arc measures $\angle A$? $\angle C$? What relation exists between these two arcs?

233. Notation for equal arcs and angles. It is often convenient to write $\angle C =$ arc AB, since the number of degrees of angle in $\angle C$ equals the number of degrees in arc AB. Also, $\angle R = \frac{1}{2}$ arc KL.

Theorem 20

234. *If two vertical angles are formed by two intersecting chords, each angle is measured by one half the number of degrees in the sum of their intercepted arcs.*



Given the chords *AB* and *CR* intersecting in the point *O*.

To prove that* $\measuredangle 1$ *and* 2 *are each measured by* $\frac{1}{2}$*(arc AC + arc BR), and that* $\measuredangle 3$ *and AOR are each measured by* $\frac{1}{2}$*(arc AR + arc BC).

Method. Draw chord *BC*. Then $\angle 1 = \angle 2 = \angle B + \angle C = \frac{1}{2}$ (arc AC + arc BR). In a similar manner, $\angle 3 = \frac{1}{2}$ (arc AR + arc BC).

Proof

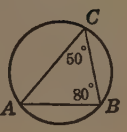
1. Draw a chord joining two ends of the given chords, as BC.	1. § 51, Postulate 1.
2. $\angle 1 = \angle 2 = \angle B + \angle C$.	2. §§ 48, 102.
3. $\angle B = \frac{1}{2}$ arc AC, and $\angle C = \frac{1}{2}$ arc BR.	3. § 229.
4. $\therefore$ $\angle 1$ or $\angle 2 = \frac{1}{2}$(arc AC + arc BR).	4. Steps 2 and 3.

Draw BR or AC. Then in like manner it may be proved that $\angle 3$ is measured by $\frac{1}{2}$(arc AR + arc BC).

QUERY. If a chord in the figure of § 234 bisects $\angle 1$ and $\angle 2$, does it bisect the arc BR and the arc AC?

EXERCISES ON THEOREMS 19 AND 20

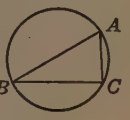
1. Two angles of an inscribed triangle are 50° and 80° respectively. Find the number of degrees in each of the arcs intercepted by the sides of the triangle.



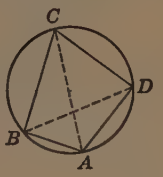
2. AB, BC, and CA are the sides of an inscribed triangle. The arc AB contains 135°, and the arc BC is twice the arc AC. Find each angle of the triangle.

3. Two sides of an inscribed triangle subtend arcs which are one fifth and one eighth of the circle respectively. Find the angles of the triangle.

4. An inscribed angle B of the right triangle ABC contains 30°. Prove that the chord AC equals the radius of the circle.



5. Four angles of an inscribed quadrilateral are 111°, 93°, 69°, and 87°, and one angle between the diagonals is 80°. Find the number of degrees in the arcs intercepted by the sides of the quadrilateral.

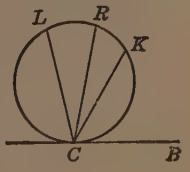


Introduction to Theorems 21, 22, and 23

EXERCISES

Use a circle equal to the protractor circle.

1. Draw a circle and a tangent CB touching the circle at C. Then draw chords CK, CR, and CL. Measure $\angle KCB$ and arc KC, $\angle RCB$ and arc RC, and $\angle LCB$ and arc LRC. Compare the number of degrees in each angle with the number in its intercepted arc.

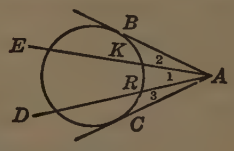


2. Repeat Exercise 1 with a new figure.

3. Complete the following statement so that it will agree with the result of Exercises 1 and 2:

The number of degrees in an angle formed by a tangent and a chord drawn from the point of contact is equal to ____ the number of degrees in its intercepted arc.

4. Draw a circle, and from an outside point A draw two tangents and two secants. Then measure $\angle 1$, $\angle 2$, and $\angle 3$ and their intercepted arcs. Compare the number of degrees in each angle and the number in the corresponding intercepted arcs. What relation exists? Measure $\angle BAC$ and arcs BC and BDC. What relation exists here?



5. Repeat Exercise 4 with a new figure.

6. Complete the following statement in accord with the results of Exercises 4 and 5:

The number of degrees in the angle between two secants, two tangents, or a tangent and a secant which intersect outside the circle is equal to ____ of the number of degrees in the ____ of their intercepted arcs.

7. Draw a circle and two parallel secants cutting it. Measure the arcs intercepted by the parallels.

8. Repeat Exercise 7 with two lines one of which is a tangent and one a secant.

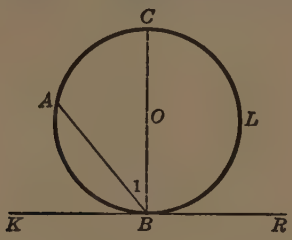
9. Repeat Exercise 8 having two tangents for the parallel lines.

10. Complete the following statement in accordance with the results of Exercises 7 to 9:

If two arcs of a circle are intercepted by parallel lines, the arcs are ____.

Theorem 21

235. *If an angle is formed by a tangent and a chord drawn from the point of contact, the angle is measured by one half the number of degrees in its intercepted arc.*



Given KR **tangent to** $\odot O$ **at** B **and chord** AB.

To prove that $\angle ABK$ ***is measured by*** $\frac{1}{2}$ ***arc*** AB ***and that*** $\angle ABR$ ***is measured by*** $\frac{1}{2}$ ***arc*** ACB.

Method. Draw BC, the diameter from the point of tangency, and prove that $\angle ABK = \text{rt. } \angle KBC - \angle 1$ and is measured by $\frac{1}{2}$ arc $BAC - \frac{1}{2}$ arc AC.

Proof

1. Draw diameter BC.	1. § 51, Postulate 1.
2. $\angle 1 = \frac{1}{2}$ arc AC.	2. § 229.
3. Rt. $\angle KBC = \frac{1}{2}$ arc BAC.	3. Both 90°.
4. $\therefore \angle KBA = \frac{1}{2}$ (arc BAC − arc AC) $= \frac{1}{2}$ arc AB.	4. Steps 2 and 3.

Similarly, $\angle ABR = \frac{1}{2}(\text{arc } AC + \text{arc } CLB) = \frac{1}{2}$ arc ALB.

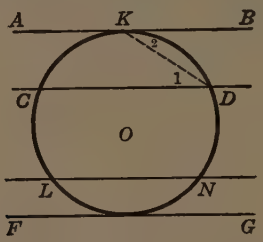
EXERCISES

1. If, in the figure of Theorem 21, $\angle ABK = 42°$, how many degrees are there in arc ALB?

2. Prove Theorem 21 if chord AB is a diameter.

Theorem 22

236. *If two arcs of a circle are intercepted by two parallel lines, the arcs are equal.*



Given $\odot O$, **with tangent** $AB \parallel$ **to secant** CD.

To prove that arc $KC =$ ***arc*** KD.

Method. Join the point of tangency of AB to either point of intersection of the circle and the secant. Then show that the two alternate interior angles formed are equal and that the arcs which measure them are also equal.

Proof

1. Draw KD. $\angle 1 = \frac{1}{2}$ arc KC.	1. § 229.
2. $\angle 2 = \frac{1}{2}$ arc KD.	2. § 235.
3. $\angle 1 = \angle 2$.	3. § 91.
4. $\therefore$ arc $KC =$ arc KD.	4. § 50, Axiom 1.

A similar proof holds if both parallels are secants. If both parallels are tangents, a secant parallel to both can be drawn and the foregoing proof applied.

QUERY 1. Is the following statement a converse of Theorem 22? If two lines intercept equal arcs of a circle, the lines are parallel.

QUERY 2. Is this statement true? If possible, draw a figure in which the statement of Query 1 does not hold true.

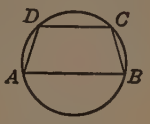
QUERY 3. Can more than one figure be drawn for Query 2?

EXERCISES

1. Prove Theorem 22 when both parallels are tangents.

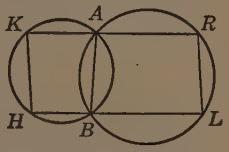
2. Prove Theorem 22 when both parallels are secants.

3. AD and BC are the nonparallel sides of the inscribed trapezoid $ABCD$. Prove that arc ABC and arc DAB are equal, and also that arc ADC and arc BCD are equal.



4. The nonparallel sides of an inscribed trapezoid are equal, and the diagonals are equal.

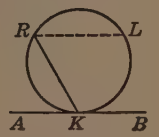
5. Two circles intersect in A and B. Through A and B parallels KAR and HBL are drawn, points K and H being on one circle and R and L on the other. Prove that the chords AB, KH, and RL are equal.



6. Two chords of a circle perpendicular to a third at its extremities are equal. Prove.

7. In the adjacent figure, RL is parallel to the tangent AB. If $\angle AKR = 64°$, how many degrees are there in arc RL?

8. If, in the figure, $\angle AKR = \frac{1}{4} \angle RKB$, how many degrees are there in arcs KR, RL, and LK?



9. If, in the adjacent figure, arc $RL = \frac{1}{2}$ arc KL, how many degrees are there in $\angle AKR$ and $\angle BKR$?

10. If a chord of a circle bisects the angle between a tangent and a chord drawn from the point of contact, does it bisect the intercepted arc? Prove.

11. If chords AB and AC of a circle make equal angles with the tangent at A, they are equal. Prove.

Theorem 23

237. *The angle between two secants, or two tangents, or a tangent and a secant which intersect outside the circle is measured by one half the number of degrees in the difference of the intercepted arcs.*

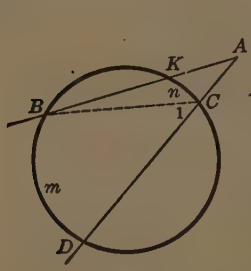


Fig. 1

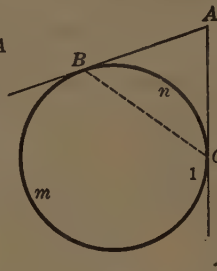


Fig. 2

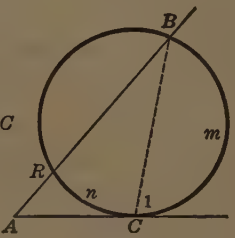


Fig. 3

Given $\angle A$ formed (*a*) by two secants, (*b*) by two tangents, (*c*) by a tangent and a secant.

To prove that $\angle A$ *is measured by* $\frac{1}{2}$(*arc* m − *arc* n).

Method. Draw BC, and prove $\angle A = \angle 1 - \angle B$ and is therefore measured by $\frac{1}{2}$(arc m − arc n).

Proof

1. In each case draw BC.	1. § 51, Postulate 1.
2. $\angle 1 = \angle A + \angle B$.	2. § 102.
3. $\therefore \angle A = \angle 1 - \angle B$.	3. § 50, Axiom 4.
4. $\angle 1 = \frac{1}{2}$ arc m.	4. § 229 in (*a*), § 235 in (*b*) and (*c*).
5. $\angle B = \frac{1}{2}$ arc n.	5. § 229 in (*a*) and (*c*), § 235 in (*b*).
6. $\therefore \angle A$ is measured by $\frac{1}{2}$(arc m − arc n).	6. Steps 3, 4, and 5.

238. ***Corollary.*** *The angle between two tangents to a circle is measured by the supplement of the smaller of the two intercepted arcs.*

Proof. In Fig. 2, $\angle A = \frac{1}{2}(m - n)$. But $m = 360° - n$. $\therefore \angle A = \frac{1}{2}[(360° - n) - n] = 180° - n$.

EXERCISES

1. If the arc KC (Fig. 1, § 237) is 34° and the arc BD is 92°, find the number of degrees in $\angle A$.

2. If the arc KC (Fig. 1, § 237) is 30° and $\angle A$ is 62°, find the number of degrees in the arc BD.

3. If the minor arc BC (Fig. 2, § 237) is one third of the major arc BC, find the number of degrees in $\angle A$.

4. If $\angle A$ (Fig. 3, § 237) contains 54° and the arc BC is four times the arc CR, find the number of degrees in the arc CR and in the arc BC.

5. If a secant bisects the angle between two tangents to a circle, does it bisect each arc intercepted by the tangents? Prove.

MISCELLANEOUS EXERCISES ON CIRCLES

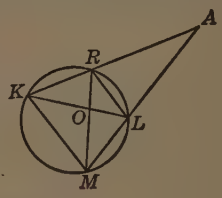
1. The chords AB and CD intersect at O. The $\angle AOC$ is 40° and the arc DB is 52°. Find the arc AC.

2. The parallel sides of an inscribed trapezoid intercept arcs of 90° and 112° respectively. Find its angles.

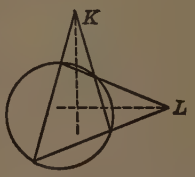
3. $ABCD$ is an inscribed quadrilateral. The arc AB is 46° and the arc BC is 96°. The arc DC is one fourth of the arc AD. Find the number of degrees in each angle of the polygon.

4. An inscribed angle A intercepts an arc y, and a central angle B in the same circle intercepts an arc x. If $x + y = 100°$ and $\angle B$ is 10° more than three times $\angle A$, find $\angle A$ and $\angle B$.

5. KR and ML, two sides of the inscribed quadrilateral $KRLM$, when produced meet in A. The diagonals of the quadrilateral meet at O. The arc RL is 70°, the arc ML is 80°, and the arc MK is 130°. Find the number of degrees in $\angle A$, $\angle RLM$, $\angle KOR$, and $\angle ARL$.



6. Assign numeric values to the angles of an inscribed quadrilateral such that its opposite sides produced will meet in K and L respectively. Then bisect $\angle K$ and $\angle L$ and determine the angle between the bisectors.



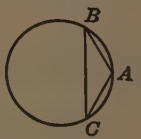
7. A chord of a circle equals the radius. Show that the minor arc of the chord is 60°.

8. Chord AB has an arc of 120°. The tangent at B is drawn and a perpendicular from A to the tangent meets it at C.

Prove $BC = \frac{1}{2} AB$.

9. AB is a diameter, and the chord AK is equal to the radius. Draw BK.

Prove that $\angle A$ is 60° and that $\angle B$ is 30°.



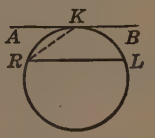
10. Chords AB and AC are equal to the radius. Draw BC and determine the number of degrees in each angle of $\triangle ABC$.

11. If the two nonparallel sides of a trapezoid are equal, its vertices are on a circle. Prove.

12. AB, a chord of a circle, is the base of an isosceles triangle whose vertex C is without the circle and whose equal sides intersect the circle in D and E respectively.

Prove $CD = CE$.

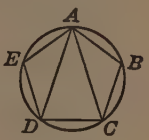
13. A is the center of a circle and K is any point outside the circle. R is the mid-point of AK. A circle with R as the center and RA as the radius cuts the first circle in H and L.



Prove that KH and KL are tangent to the first circle.

14. A tangent at the mid-point of an arc of a circle is parallel to the chord of the arc.

15. The perimeter of an inscribed equilateral triangle is equal to one half the perimeter of the circumscribed equilateral triangle. Prove.



16. The diagonals of an inscribed equilateral polygon of five sides are equal. Prove.

17. The bisectors of two opposite angles of an inscribed quadrilateral meet the circle in A and B.

Prove that AB is a diameter of the circle.

18. ABC is an inscribed equilateral triangle. DE joins the middle points of arcs BC and AC.

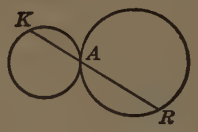
Prove that DE is trisected by the sides BC and AC.

19. ABC is an inscribed isosceles triangle. Tangents are drawn at A, B, and C forming a circumscribed triangle.

Prove that the circumscribed triangle is isosceles.

20. If one of the equal angles of the inscribed triangle in the preceding exercise equals the vertex angle of the circumscribed triangle, find the number of degrees in it.

21. Any secant through the point of contact of two tangent circles subtends arcs of which the two in one circle contain the same number of degrees respectively as the two in the other circle. Prove.



HINT. Draw the common tangent.

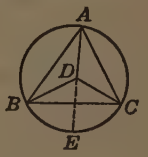
22. A secant through the point of contact, A, of two tangent circles cuts them in K and R respectively.

Prove that the radii drawn to K and R are parallel.

23. AB, a diameter, is extended half its length to K. Tangents from K cut the tangent at A in R and L respectively.

Prove that $\triangle KRL$ is equilateral.

24. ABC is an inscribed triangle. The bisectors of $\measuredangle A$, B, and C meet in D. AD produced meets the circumference in E.

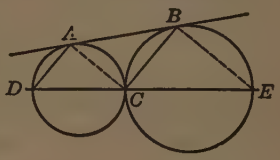


Prove $DE =$ chord CE.

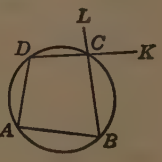
25. Two circles intersect at A and B, and a tangent to each circle is drawn at A meeting the circles at R and S respectively.

Prove that $\measuredangle ABR$ and ABS are mutually equiangular.

26. A common external tangent is drawn to two circles which are externally tangent. Show that the chords drawn to the points of tangency from the points where the line through the centers cuts the circles are parallel in pairs.

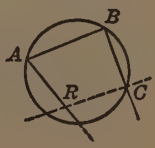


27. How are the exterior angles of an inscribed polygon measured? Consider $\angle LCD$. State a theorem which tells by what arcs such an angle is measured. Prove.



28. $ABCD$ is an inscribed polygon. The chords AB, BC, and CD subtend arcs which are one third, one fourth, and one fifth of the circle respectively. Find the number of degrees in each exterior angle of $ABCD$.

29. A, B, and C, three points on a circle, and R, a point within, are the vertices of quadrilateral $ABCR$. Prove that $\angle A$ is less than the exterior angle at C.

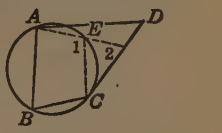


EXERCISES ON CIRCLES THROUGH FOUR POINTS

1. Prove that the opposite angles of an inscribed quadrilateral are supplementary.

2. Prove that if the opposite angles of a quadrilateral are supplementary, a circle will pass through its vertices.

HINT. Construct a circle passing through A, B, and C. If D falls outside the circle, then $\angle 1 > \angle 2 > \angle D$ and $\angle B + \angle 1 = 180°$, etc. Prove in like manner that D cannot fall inside the circle.



3. Prove by Exercise 2 that a circle can be circumscribed about a rectangle.

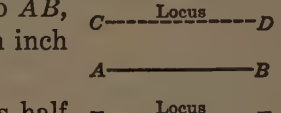
4. Do the vertices of a square lie on a circle? of a rhombus? of a parallelogram? Prove.

5. In the trapezoid $ABCD$, $\angle A = \angle B$, and $\angle C = \angle D$. Do the points A, B, C, and D lie on a circle? Prove.

6. K is a point outside $\triangle ABC$. KR, KL, and KM are the perpendiculars to AB, CA, and BC respectively, produced if necessary. Prove that a circle will pass through the four points A, R, L, and K, another through the four points C, K, L, and M, and a third through the four points B, R, K, and M. Locate the center and radius of each circle.

Loci

239. Examples of loci. Where are all the points whose distance from a given line AB is half an inch? Evidently they are on two lines each parallel to AB, one on each side of AB and half an inch from it. Here



(*a*) every point on CD or EF is half an inch from AB;

(*b*) every point half an inch from AB is on either CD or EF.

Where are all the points half an inch from point A? It is evident that every point half an inch from A lies on a circle whose radius is half an inch and whose center is A. Again,

(*a*) every point on the circle is half an inch from A;

(*b*) every point half an inch from A is on the circle.

Each of the two preceding questions deals with what is called a *locus* (plural *loci*, pronounced lō′sī), a Latin word meaning "place."

240. Locus. A *locus* is a figure containing all the points, and only those points, which fulfill a given requirement.

For example, the first locus above consists of two lines. Although every point in either of these lines satisfies the required condition, neither line alone can properly be called the locus, because it does not contain all the points which fulfill the given requirement.

Expressed in another way, a locus is the complete path of a point which moves in such a way that its every position fulfills a given geometric condition.

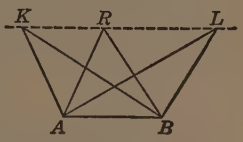
Thus, every point in a railway car moving on a piece of straight level road moves in a line parallel to the rails. Hence the locus of the path of a particular point on the car (except on the wheels) is a horizontal straight line.

Also, every point on a moving elevator is going up or down in a line perpendicular to the ground. Here the path of a particular point on the elevator is a vertical straight line.

A locus may be a very complicated curve, but the only loci studied here will be figures composed of straight lines or circles or some combination of the two. Therefore, in all the exercises on loci given in this text, if three points of the required locus are *accurately* located, it can be decided at once whether the locus is a straight line or a circle.

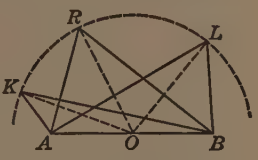
EXAMPLES

1. Triangles having the same base and altitudes are constructed on one side of a fixed base AB. Find the locus of their vertices.



Solution. Draw accurately three triangles on the given base having equal altitudes. The locus is at once seen to be a straight line parallel to AB and at a distance from it equal to the given altitude.

2. Triangles are constructed on one side of a given base AB with medians equal to a given line m. Find the locus of their vertices.



Solution. Draw accurately three triangles having a median equal to m. The locus of the vertices of all such triangles can at once be seen to be a semicircle whose radius is m and whose center is the mid-point of AB; for since the points K, R, and L are certainly not on a straight line, they must be on a circle.

QUERY 1. What is the locus in Example 1 if the triangles are on both sides of AB?

QUERY 2. What is the locus in Example 2 if the triangles are on both sides of AB?

EXERCISES

1. Draw the locus of points equidistant from two parallel lines which are 2 in. apart.

2. A rectangle is 4 in. by 3 in. Draw the locus of points within the rectangle half an inch distant from a side. What kind of figure is the complete locus?

3. The distance between two parallel lines AB and CD is 3 in. Draw the locus of the points lying between them which are twice as far from AB as from CD.

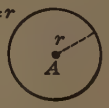
4. Draw the locus of a point 1 in. from a fixed point A.

5. A fixed circle has a radius of 1 in. Draw the locus of points half an inch from the circle and inside it.

6. A fixed circle has a radius of 2 in. Draw the locus of all the points whose distance from the circle is half an inch.

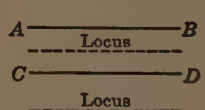
241. Fundamental locus theorems. There are a few simple loci which are so essential in solving problems in loci that they may be regarded as fundamental theorems. These are the following:

1. The locus of a point at a given distance from a given point is a circle whose center is at the point and whose radius is the given distance.

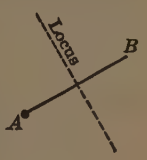


2. The locus of a point equidistant from two parallels is a line parallel to both midway between them.

3. The locus of a point at a given distance from a straight line is two lines parallel to the given line at the given distance from it.

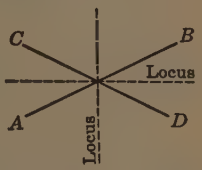


4. The locus of a point equidistant from two given points is the perpendicular bisector of the line joining the two points.



It has been proved (*a*) that every point on the perpendicular bisector of a line is equidistant from the ends of the line (§ 136), and (*b*) that every point equidistant from the ends of the line is on the perpendicular bisector of the line (§ 137).

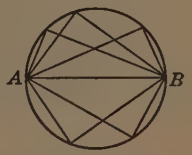
5. The locus of points equidistant from two intersecting lines is the two bisectors of the angles included by the lines.



It has been proved (*a*) that every point on the bisector of an angle is equally distant from the sides of the angle (§ 141), and (*b*) that every point equally distant from the sides of an angle lies in the bisector (§ 142).

6. The locus of points at a distance r from a fixed circle of radius R and center A is two circles with center A and radii $R + r$ and $R - r$ respectively.

7. The locus of the vertex of the right angle of a right triangle which has a fixed hypotenuse AB is a circle with AB as a diameter.



EXERCISES ON LOCI

In the following exercises find the complete locus. It may consist of more than one line. If the locus is not apparent, locate accurately three points on it. This will determine where it is and whether it is a circle or a straight line.

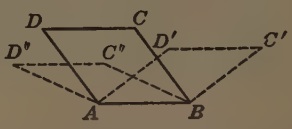
1. The distance between two parallels AB and CD is 3 in. Draw the locus of all points half as far from AB as from CD.

2. The distance between two parallels AB and CD is 2 in. Draw the locus of all points three times as far from AB as from CD.

3. The distance between points A and B is 4 in. Draw the locus of all points equidistant from A and B.

4. Lines AB and CD intersect at O. Draw the locus of a point whose distance from AB equals its distance from CD.

5. The side AB of rhombus $ABCD$ is fixed. Draw the locus of the vertex D of all the rhombuses having AB for one side.



6. Point K is outside a fixed line AB. Lines are drawn from K to points on AB. Draw the locus of their mid-points.

7. AB and CD are the parallel sides of a trapezoid. Lines are drawn from any point in AB to any point in CD. Draw the locus of the mid-points of these lines.

8. How many circles may touch a fixed line AB at a point K on the line? Draw the locus of their centers.

9. How many circles of radius 1 in. can be drawn tangent to a fixed line AB? Draw the locus of the centers of all these circles. How many lines are there in the locus?

10. Draw the locus of the mid-points of the radii of a given circle.

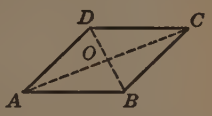
11. How many circles may pass through two fixed points which are 3 in. apart? Draw the locus of their centers.

12. Lines are drawn parallel to side AB of $\triangle ABC$ and terminating in sides AC and BC. Draw the locus of the mid-points of these lines.

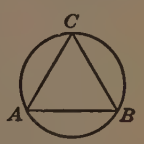
13. Lines parallel to the base of a given trapezoid terminate in the nonparallel sides. Draw the locus of their mid-points.

14. Draw the locus of the vertices of all the right angles whose sides pass through two fixed points A and B. How does this locus differ from that of § 241, 7?

15. On one side of a fixed line AB as a base how many rhombuses are possible? Draw the locus of the point of intersection of their diagonals.



16. ABC is an inscribed equilateral triangle. Inscribe another triangle on the base AB having the angle opposite equal to 60°. How many such triangles are possible? How many are possible on the opposite side of AB? Is the locus a complete circle?



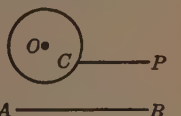
17. A coin $1\frac{1}{2}$ in. in diameter is moved around a fixed coin $\frac{7}{8}$ in. in diameter, the two coins constantly touching each other and remaining in the same plane. What is the path of the center of the larger coin? Draw the locus of the path.

18. A hay barn, 40 ft. by 80 ft., stands in a meadow. A horse is tied outside the barn to one corner by a rope 100 ft. long. Draw a diagram showing the boundary of the area over which he can graze.

19. Chords are drawn through A, a fixed point on a circle. Draw the locus of their mid-points.

HINT. Draw three or more chords and locate accurately their mid-points. Are these points in a straight line or a curved one? Draw several other chords and locate their mid-points.

20. A line CP of given length moves so that it is always parallel to a fixed line AB and so that C remains on circle O. Draw the locus of point P.

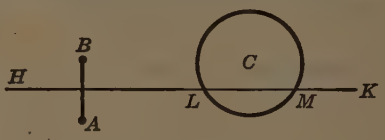


242. Intersection of loci. In some locus problems the point considered must satisfy two conditions. Then each condition determines a locus, and the intersection of the two loci locates the required point or points.

EXAMPLE

Find all points equidistant from A and B and one-half inch from C.

Solution. Construct HK, the perpendicular bisector of AB. With C as center describe a circle with a half-inch radius. This circle cuts KH in points L and M.



Proof. All points equidistant from A and B are on HK, and all points half an inch from C are on the circle. $\therefore$ L and M are the points.

Discussion. No points could satisfy both conditions if C were more than half an inch from HK. There would be only one point if C were just half an inch from HK. If C is less than half an inch from HK there are two points, as above.

EXERCISES ON INTERSECTION OF LOCI

1. Two points, A and B, are 4 in. apart. Find all the points 2 in. from A and 3 in. from B.

HINT. Where are all the points 2 in. from A? 3 in. from B? Where, then, are the required points?

2. A point A is 3 in. from the line BC. Locate all the points 2 in. from point A and the line BC.

HINT. Where are all the points 3 in. from line BC? those 2 in. from point A? Where, then, are the required points?

3. If the line AB is not perpendicular to line FG, locate the points on FG equidistant from A and B.

4. Locate all points on a line FG which are equidistant from two parallel lines AB and CD.

5. Locate all the points equidistant from two points A and B and from two intersecting lines FG and KL.

6. A point K is 1 in. from a given line AB. Locate all the points which are 2 in. from K and 3 in. from AB.

7. A circle lies between two parallel lines. Locate all points on the circle equidistant from the parallels.

8. The sides of a triangle, AB, BC, and CA, are 4, 5, and 6 centimeters respectively. Locate all points 2 centimeters from A and equidistant from B and C.

In the following exercises state the conditions under which no point, one point, or several points may be found:

9. Locate all points on a line AB 1 in. distant from a fixed point K.

10. Locate all points which are a given distance from a fixed point and a fixed line.

11. Locate all points on a given line equidistant from two fixed points.

12. Locate all points on a line FG which are equidistant from two intersecting lines AB and CD.

13. Three lines intersect in points A, B, and C. Locate all points equidistant from A and B which are also equidistant from lines AC and BC.

14. Locate all points equidistant from two given intersecting lines and equidistant from two fixed points.

15. Locate all points 1 in. from a given circle whose radius is 2 in. and 4 in. from a given line.

16. A triangle is to have a fixed base AB and a median of given length m on AB. Locate its vertex.

17. Locate all points at a given distance from a fixed line AB and equidistant from two given points.

18. A triangle is to have a fixed base AB, a given altitude h, and a given median m on AB. If $m > h$, locate the vertex.

19. Locate all centers of circles of given radius which touch a given line and pass through a given point.

243. The straight line and its equation. It was shown in elementary algebra that equations like $x + y = 6$, $x - y = 4$, and $2x + 3y = 12$ were each satisfied by innumerable pairs of values of x and y. For example, the pairs of values $x = 0$ and $y = 4$, $x = 6$ and $y = 0$, $x = 3$ and $y = 2$ all satisfy $2x + 3y = 12$. It was shown that with an x-axis and a y-axis perpendicular to each other, and with a given unit of distance, any pair of numbers x and y represented a point. The x value was the distance of the point from the y-axis, and the y value was its distance from the x-axis. Moreover, when three or more pairs of values of x and y which satisfied an equation like $2x + 3y = 12$ were plotted, the points were found to be in a straight line. This made it clear that an equation was a *locus*. It was the locus of all points whose x and y distances satisfied the equation. The study of this relation resulted in establishing a connection between algebra and geometry, for it showed that a first-degree equation in x and y could be represented by a geometric figure, namely, a straight line.

EXERCISES

1. With reference to two axes X and Y, answer the following questions:

(*a*) What is the locus of a point for which $x = 2$? $x = -4$?

(*b*) What is the locus of a point for which $y = 3$? $y = -5$?

2. Locate the following points: (*a*) (3, 4), (*b*) (3, -4), (*c*) (4, -2), (*d*) (-2, -4). (Here the first number named is the x distance and the second the y distance.)

3. Draw a rectangle whose vertices are (2, 2), (8, 2), (2, 6), and (8, 6).

4. Draw a triangle whose vertices are (0, 0), (0, 8), and (4, 5).

Construct the graph of the following equations:

5. $x + y = 6$.

6. $x - y = 4$.

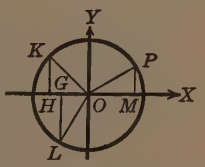
7. $2x + 3y = 12$.

8. $x - 2y = 10$.

9. The sides of a triangle are $x + y = 6$, $x - y = 6$, and $5x = y + 6$. Construct the graph of the sides of the triangle.

244. The circle and its equation. The straight line, as we have seen, has an equation. Similarly, the circle has an equation which is easily obtained.

Let O, the point of intersection of the axes, be the center of the circle and r be its radius. For P, any point on the circle, $OM = x$ and $PM = y$. As in arithmetic, $\overline{OP}^2 = \overline{OM}^2 + \overline{PM}^2$. That is, $x^2 + y^2 = r^2$. This same equation holds for point K, where $x = HO$ and is negative and $y = KH$ and is positive. Similarly, for point L, where x and y are both negative, $x^2 + y^2 = r^2$. That is, this right-triangle relation holds for every point on the circle, and the equation given merely expresses this relation. This equation, $x^2 + y^2 = r^2$, is called the equation of the circle whose center is at the origin and whose radius is r, because every point on the circle has x distances and y distances that satisfy the equation, and conversely.



EXERCISES

1. What are the radii of the circles whose equations are (*a*) $x^2 + y^2 = 4$, (*b*) $x^2 + y^2 = 25$, (*c*) $x^2 + y^2 = 9$?

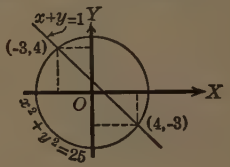
2. Draw the circle whose equation is $x^2 + y^2 = 9$.

The relation between the lines themselves and their equations can be illustrated as follows:

EXAMPLE

What points are on the line $x + y = 1$ and the circle $x^2 + y^2 = 25$?

Solution. The lines corresponding to these two equations are shown in the adjacent figure. The points on both lines are at their intersection points, $(-3, 4)$ and $(4, -3)$. This result can be confirmed by algebra by merely solving the equations simultaneously.



$$x^2 + y^2 = 25, \qquad (1)$$

$$x + y = 1. \qquad (2)$$

$$x = 1 - y. \qquad (3)$$

Substituting in (1), $(1 - y)^2 + y^2 = 25$,

or $2y^2 - 2y + 1 = 25$.

Simplifying, $y^2 - y - 12 = 0$.

Factoring, $(y - 4)(y + 3) = 0$.

$\therefore y = 4$ and -3.

Substituting in (3), $x = -3$ and 4.

Therefore $(-3, 4)$ and $(4, -3)$ are the two pairs of roots which satisfy the equations. These are the x and y distances of the two points in which the straight line cuts the circle, as shown in the graph.

EXERCISES

1. Draw the graphs of $x^2 + y^2 = 25$ and $2y - x = 5$, and find the x and y distances of the points of intersection.

2. Determine by drawing graphs whether the straight line $3x + 4y = 25$ is tangent to the circle $x^2 + y^2 = 25$.

245. The relation between algebra and geometry. It is desirable in this text to consider only circles and straight lines in problems with graphs, yet it is interesting to know that any algebraic equation in x and y can be graphed and hence represents a line of some sort. Thus an equation in x and y defines in algebraic language a geometric line or locus. The geometric property which points on this line possess is that the x and y values of each point satisfy the algebraic equation. Further, the more difficult step of finding the equation of any geometric line is possible. In this way the problems of algebra can be expressed and interpreted by geometry, and vice versa. The method of doing this, which involves the use of axes, is called graphics and is part of the larger subject of analytic geometry.

The eminent mathematician Lagrange said: "As long as algebra and geometry traveled separate paths their advance was slow and their applications limited. But when these two sciences joined company they drew from each other fresh vitality and thenceforward marched at a rapid pace towards perfection." It is to Descartes (1596–1650) that we owe the application of algebra to geometry, — an application which has furnished the key to the greatest discoveries in all branches of mathematics.

Summary of Construction Work

246. Postulates of construction. The possibility of three simple operations, which underlie all the construction work of elementary geometry, is assumed in Postulates 1, 3, and 9 of § 51.

247. Instruments used in constructions. Elementary geometry deals with figures composed only of straight lines or of circles or of the two combined. Therefore only two instruments are needed: the straightedge for drawing straight lines and the compasses for drawing circles. Construction work is consequently limited to the two instruments actually required.

It is to be noted, however, that in any *drawing* work, even in drawing the figures used in geometry, we may work with any instruments we please. In fact the ruler, marked in inches or centimeters, and the protractor have already been used and found helpful.

248. Problems of construction. Methods have been given for constructing some of the figures so far needed in the demonstrations of the theorems and exercises. The construction of the other figures studied can be accomplished by means of the eleven constructions already given. A greater variety of construction exercises, involving these and one more fundamental construction, will now be taken up. It is true that the methods underlying construction work are only theoretically exact, since, however sharp our pencil, we cannot draw a geometric line nor can we draw with the best ruler a line which is perfectly straight. The resulting figures are accurate enough, however, for many practical purposes, and — what is very important — they show that certain figures needed in the demonstrations are possible. Moreover, the exercises in construction furnish us with a kind of experience by which our knowledge of geometry can be extended and perfected to a greater extent, perhaps, than merely by the work of formal demonstration as found in the theorems.

249. The fundamental constructions. The constructions previously solved are now restated, and the section where each may be found is given for convenient reference.

1. Construct a triangle given its three sides (page 5, Exercise 8).

2. Construct a perpendicular to a line at a given point in the line (§ 75).

3. Construct a perpendicular to a line from a point outside the line (§ 76).

4. Given one side and the vertex, construct an angle equal to a given angle (§ 85).

5. Construct a parallel to a line through a point not on the line (§ 99).

6. Construct the perpendicular bisector of a given line segment (§ 135).

7. Bisect a given angle (§ 140).

8. Divide a given straight line into three or more equal parts (§ 146).

9. Construct a circle which passes through three given points (§ 174).

10. Bisect a given circular arc (§ 188).

11. Construct a tangent to a circle from an outside point (§ 215).

250. General suggestions for solving a construction problem. If the solution is not apparent after a brief study proceed as follows:

1. Draw freehand (do not try to construct) as accurately as possible a figure which at least approximately satisfies the given conditions and represents the required figure. Mark the parts which correspond to the given parts in some way that will make them stand out among the other lines and angles of the figure.

2. By careful inspection of this figure try to discover a method of correctly constructing it. If a method of construction is not apparent, attempt to answer questions like the following: Can part of the figure (a triangle perhaps) be constructed from the given data? If so, can another part then be constructed? Then, finally, can the entire figure be constructed?

3. If the main difficulty seems to lie in correctly locating a certain point, consider the possibility of the point (or points) being at the intersection of (*a*) two straight lines; (*b*) a straight line and a circle; (*c*) two circles.

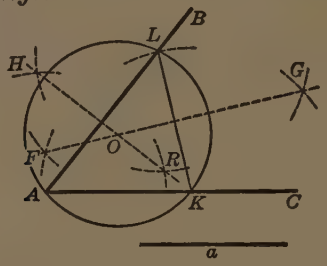
4. If the preceding steps fail, repeat them with a new figure which satisfies approximately the conditions but which differs in appearance from the first.

5. When the solution has been found, construct the required figure.

6. Discuss the conditions under which the problem has no solution, one solution, or several solutions.

Construction 3

251. *With a line of given length as a chord construct a segment such that every angle inscribed in it shall equal a given angle.*



Given the line a and $\angle BAC$.

Required to construct a segment whose chord is equal to a and such that every angle inscribed in it shall equal $\angle BAC$.

Method. With K, any point on AC, as the center and a as the radius, describe an arc cutting AB, as at L. Draw KL. Construct a circle through A, K, and L as in § 174. Then KAL is the required segment.

Proof

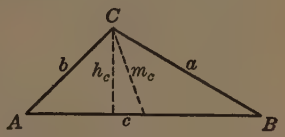
1. $KL = a$.	1. By construction.
2. $\angle LAK = \angle BAC$.	2. They are identical.
3. Any $\angle$ inscribed in segment $KAL = \angle BAC$.	3. § 231.

EXERCISES

1. On a chord 3 in. long construct a segment which shall contain an angle (*a*) of 60°, (*b*) of 45°, (*c*) of 30°.

2. If an angle inscribed in one segment of a circle contains m degrees, how many degrees are there in the angle inscribed in the remaining segment? Prove.

252. Notation for parts of a triangle. In any triangle ABC the angles are frequently denoted by the capital letters A, B, and C and the sides opposite these angles by the small letters a, b, and c.



The altitudes on the sides a, b, and c are therefore conveniently represented by h_a (read h sub a), h_b, and h_c respectively.

Similarly, the respective medians to sides a, b, and c may be denoted by m_a, m_b, and m_c.

MISCELLANEOUS CONSTRUCTION EXERCISES

1. Construct an angle of 60°.

2. Construct an angle of 45°.

3. Divide a right angle into three equal angles.

4. Construct an angle of 120°.

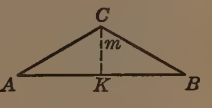
5. Construct the supplement of a given angle.

6. Construct the complement of a given acute angle.

7. Construct an isosceles triangle given the base and the altitude.

8. Construct an isosceles triangle given the base and the vertical angle.

HINT. Given AB and $\angle C$. Let CK be perpendicular to AB. What relation exists between $\angle m$ and $\angle B$? How can $\angle B$ be constructed?



9. Construct a right triangle given the hypotenuse and an acute angle.

10. Construct a right triangle given one side and the acute angle opposite.

11. Construct a right triangle given the two sides about the right angle.

12. Construct a right triangle given the hypotenuse and another side.

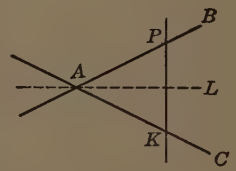
HINT. Construct a semicircle on the hypotenuse AB.

13. Construct an isosceles triangle given the base and the altitude on one of the equal sides.

HINT. Suppose $\triangle ABC$ is the completed triangle. How can the part ABK be constructed?

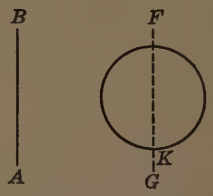
14. Through a given point draw a line making a given angle with a given line.

15. Draw a line through a fixed point cutting the sides of a fixed angle and forming an isosceles triangle.



HINT. Suppose BAC is the given angle, P the given point, and PK the required line. How is AL drawn? What angle does PK make with AL?

16. Construct a tangent to a fixed circle which shall be perpendicular to a fixed line.



HINT. How is FG drawn? Will a tangent at K be perpendicular to AB?

17. Construct a tangent to a fixed circle which shall be parallel to a fixed line.

18. Construct the bisector of the angle between two lines without using their point of intersection.

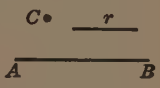
19. Construct a circle having its center in a given line and passing through two fixed points.

HINT. What is the locus of the center of all circles passing through two given points?

20. With a given radius construct a circle tangent to each of two intersecting lines.

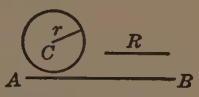
21. With a given radius construct a circle tangent to a given line and passing through a given point.

HINT. Where are all points whose distance from AB is r? Where are all points whose distance from C is r?



22. Construct a circle of given radius tangent to a fixed circle and a fixed line.

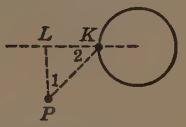
HINT. Where are all the points whose distance from AB is R? How far is the center of the required circle from point C?



23. Construct a circle tangent to each of two given lines and touching one of them at a given point.

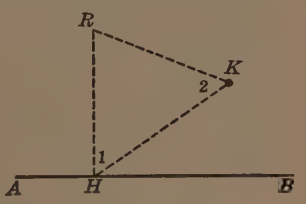
24. Construct a circle that will pass through a given point and will be tangent to a given circle at a given point.

HINT. Study the adjacent figure and discover how point L may be located.



25. Construct a circle that will pass through a given point and will be tangent to a given line at a given point.

HINT. Let K be one of the given points and H the other. How can a point R be found such that $RK = RH$? What relation exists between $\angle 1$ and $\angle 2$?



Construct a triangle given the following:

26. a, b, and m_a.

27. a, b, and h_a.

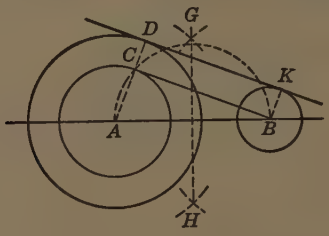
28. a, h_c, and A.

29. a, b, and h_c.

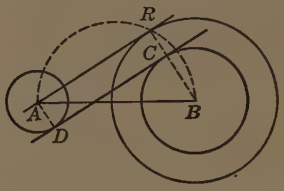
30. a, c, and m_a.

31. A, B, and h_a.

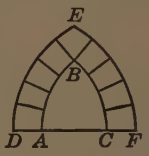
32. Construct the common external tangent to two circles.



33. Construct the common internal tangent to two circles.



34. The adjacent figure gives the outlines of a Gothic arch. The curves used in such arches are circles.



(*a*) Are the centers of the arcs BC and EF above or below the line DF?

(*b*) Is the center of arc BC on AB or AB produced?

(*c*) Have the arcs BC and EF the same center?

35. Construct a Gothic arch similar to the one above in which AC is 2 in. and B is $1\frac{1}{2}$ in. distant from AC.

REVIEW QUESTIONS

1. What is the length of the longest chord in a circle whose radius is 4 in.?

2. What are the differences between a chord, a secant, and a tangent?

3. Is a circle determined (*a*) if two points on it are known? (*b*) if the center and diameter are known? (*c*) if three points on it are known? (*d*) if four points on it are known?

4. Through a point on a circle how many chords can be drawn? how many secants? how many diameters? how many tangents?

5. If a circle is circumscribed about a right triangle, what is its radius? where is its center?

6. The ends of the diameter of a circle are joined to a point. What kind of angle do the two lines form if the point is outside the circle? inside the circle? on the circle?

7. How many chords has an arc? How many arcs has a chord?

8. How small may an inscribed angle be? How large? What limits the size of an inscribed angle?

9. How large may the angle between two secants be? the angle between two tangents?

10. What is the longest chord through a point inside a circle? the shortest one?

11. If a number of angles inscribed in the same segment are bisected, will the bisectors meet at a point? Where?

12. May a straight line have three points in common with a circle? two points? one point?

13. May two circles have three points in common? two points? one point?

14. Consider an arc, its chord, and its central angle, and complete the following:

(*a*) In equal circles, if two arcs are equal, ____.

(*b*) In equal circles, if two chords are equal, ____.

(*c*) In equal circles, if two central angles are equal, ____.

15. In the same way complete the following:
(*a*) In equal circles, if two arcs are unequal, ____.
(*b*) In equal circles, if two chords are unequal, ____.
(*c*) In equal circles, if two central angles are unequal, ____.

16. How is an angle measured (*a*) if it is a central angle? (*b*) if it is an inscribed angle? (*c*) an angle between two chords? (*d*) one between two secants? (*e*) one between two tangents?

17. The $\angle ABC$ is an inscribed angle. CB is produced to K. What arcs measure $\angle ABK$ and how?

18. Under what conditions are two inscribed angles equal? supplementary?

19. If a line is perpendicular to a chord at its mid-point does it bisect the arc of the chord?

20. If a line bisects an arc, will it bisect the chord of the arc?

21. Why are the opposite angles of an inscribed quadrilateral always supplementary?

22. Why are the opposite angles of an inscribed hexagon not supplementary?

23. How many tangents can be drawn to a circle from a point on the circle? a point inside the circle? a point outside the circle?

24. How many points determine a straight line? two parallel lines? a circle? an angle?

25. Under what conditions can two circles have no common tangent? one common tangent? two? three? four? more than four?

26. Does the bisector of an inscribed angle bisect the intercepted arc?

27. Does the bisector of the angle between two tangents bisect each of the two intercepted arcs?

28. If a circle is circumscribed about a rectangle, where is its center? What is the radius of the circle?

29. Is the line joining the centers of two circles ever equal to either of the common tangents? Is it ever less than either? greater than either?

30. If the bisector of an angle between two chords passes through the center of the circle, does it bisect the intercepted arcs? If it does not pass through the center, can it bisect the intercepted arcs?

Summary

(Black figures refer to sections)

253. Résumé of Book II. The main work of Book II is a study of the circle and certain straight lines related to it. It involves relations between central angles, chords, and arcs, both equal and unequal; properties of tangent lines and tangent circles; the measurement of central, inscribed, and other angles by means of their intercepted arcs; and the use of the properties of the circle in obtaining loci and in constructions. The work may be divided into six parts:

1. *The circle and its properties.* Fundamental assumptions, **171**; interior of a circle, **172**; contrast between a straight line and a circle, **173**; passing the circle through three points, **174**; circumference, **175**; central angle, **176**; chord, **177, 210**; semicircle, **178**; arcs, **179, 180**; tangents, **197, 202, 203, 204, 209**; secant, **198**; circumscribed and inscribed polygons, **199, 200**; concentric circles, **201**; line of centers, **205**; tangent circles, **206, 207, 208**; common

chord, 210; degree of arc, 224; relation of central angle to its intercepted arc, 225; inscribed angle, 226; segment, 227; sector, 228.

2. *Relations between central angles, chords, and arcs.* Theorems on equal central angles, 181, 183; theorems on equal arcs, 181, 183, 185, 186, 236; theorems on equal chords, 185, 186, 189, 190; theorems on unequal central angles, 182, 184; theorems on unequal arcs, 182, 184, 193, 194; theorems on unequal chords, 191, 193, 194, 195, 196; a line through the center of a circle perpendicular to a chord bisects the chord and the arcs, 187; the diameter of a circle is greater than any other chord, 191.

3. *Tangent lines and circles.* Theorems on lines tangent to circles, 211, 212, 213, 235, 236, 237, 238; theorem on circles tangent to each other, 217.

4. *Measurement of angles by arcs.* Theorems on arc and angle relations for a circle, 225, 229, 230, 231, 232, 233, 234, 235, 237, 238; arcs between parallel lines, 236.

5. *Loci involving straight lines and circles.* Definition of locus, 240; examples of loci, 239; fundamental theorems on loci, 241; intersection of loci, 242; the straight line and its equation, 243; the circle and its equation, 244.

6. *Constructions using straight lines and circles.* Postulates of construction, 246; instruments of construction, 247; fundamental constructions, 249; general suggestions for solving construction problems, 250.

EUCLID, TEACHER AND WRITER

In the field of mathematics there is no name more noted than that of Euclid (about 300 B.C.), the first professor of mathematics in the first international university in the world, the University of Alexandria, which was located in the city

founded by Alexander the Great (356–323 B.C.). The University was founded by Ptolemy (323–283 B.C.), who built its zoölogical gardens, its laboratories, and its museum and thus made Alexandria the center of the literary and scientific world.

Euclid's great reputation rests securely on his "Elements," a textbook devoted mainly to plane and solid geometry. Since 1482 over one thousand editions of his geometry have been printed, which indicates a vogue surpassing that of any novelist and makes him the most successful textbook-writer that ever lived.

The thirteen books of the "Elements" summed up in a masterly way the works of his predecessors. It seems that much of Books I and II was due to Pythagoras, of Book III to Hippocrates, of Book V to Eudoxus, of Books IV, VI, XI, and XII to later Greek writers. Books VII, VIII, and IX were devoted not to geometry but to the theory of prime and composite numbers. Though the "Elements" was prepared for use in his own classes, no original exercises for the student were included, and certain important practical theorems like that on the area of a triangle were omitted.

Alexandria was reduced about A. D. 640, but some of the learning centered there was preserved in various Syrian cities on the east coast of the Mediterranean. The translation of Euclid's "Elements" into Arabic was begun in the reign of Harun-al-Rashid (786–809) and completed under Al-Mamun (813–833). The Arabs conquered Spain in 711 and later brought Euclid's "Elements" with them, and here for over three centuries they guarded it carefully. About the year 1120 it was translated into Latin, and by 1516 five different editions had been printed.

The "Elements" arrived in Europe by still another route. Constantinople fell into the hands of the Turks in the year 1453, and many educated Greeks fled to Italy, taking with them numerous valuable manuscripts of Greek literature — Euclid's "Elements" among them. The first English translation of the "Elements" was made by Sir Henry Billingsley

in 1570. The page of the Billingsley translation shown below gives Euclid's proof of the theorem about the vertical angles formed by intersecting straight lines. It is interesting to compare Euclid's proof with that on page 31 of this text.

Of Euclid the man little is known, though one story has come down to us. King Ptolemy asked Euclid if there was no short way through the "Elements," and Euclid replied, "There is no royal road through geometry."

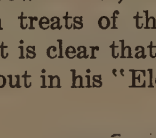
Euclid's "Elements" was first used at Harvard in 1726 and at Yale about six years later. Euclid wrote other books which have not come down to us, one of which was on conic sections, a subject which treats of the ellipse, the parabola, and the hyperbola. So it is clear that Euclid knew much more mathematics than he put in his "Elements."

The first Booke

Equall vnder the line B D: namely, that the whole shalbe equall to the part: which is also impossible. Wherefore C D is one right line: which was required to be proued.

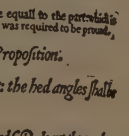
The 8. Theoreme. The 15. Proposition.

If two right lines cut the one the other: the bed angles shalbe equal the one to the other.



Demonstration.

VPpose that these two right lines A B and C D, do cut the one the other in the point E. Then I say, that the angle A E C, is equall to the angle D E B. For forasmuch as the right line A E. standeth vpon the right line D C, making these angles C E A, and A E D: therefore by the 13. proposition the angles C E A, and A E D, are equall to two right angles. Againe forasmuch as the right line D E, standeth vpon the right line A B, making these angles A E D, and D E B, therfore (by the same proposition) the angles A E D, and D E B, are equall to two right angles. and it is proued, that the angles C E A, and A E D, are also equall to two right angles. VVherfore the angles C E A, and A E D, are equall to the angles A E D, and D E B. Take away the angle A E D, which is common to them both. VVherefore the angle remayning C E A, is equall to the angle remayning D E B. And in like sort may it be proued, that the angles C E B, and D E A, are equall the one to the other. If therefore two right lines cut the one the other, the bed angles shalbe equall the one to the other: which was required to be demonstrated.



Thales Milesius the first inuenter of this proposition.

Thales Milesius the Philosopher was the first inuenter of this Proposition: witnesseth *Eudemius*, but yet it was first demonstrated by *Euclide*. And in it there is no construction at all. For the exposition of the thing geuen, is sufficient enough for the demonstration.

Bed Angles.

Bed Angles, are opposite angles, caused of the intersection of two right lines and are so called, because the heddes of the two angles are ioyned together in one pointe.

The conuerse of this proposition after Peletarius.

The conuerse of the proposition of Peletarius.

If fower right lines being drawen from one point, do make fower angles, of which the opposite angles are equall: the two opposite lines shalbe drawen directly, and make one right line.

Suppose that there be fower right lines A B, A C, A D, and A E, drawen from the poynt A, making fower angles at the point A: of which let the angle B A C be equalle to the angle D A E, and the angle B A D to the angle C A E. Then I say, that B E and C D are only two right lines: that is, the two right lines B A and A E are drawen directly and

BOOK III

RATIO, PROPORTION, AND SIMILAR FIGURES

When you can measure what you are speaking about and express it in numbers, you know something about it, but when you cannot measure it, when you cannot express it in numbers, your knowledge is of the meager, unsatisfactory kind. — KELVIN

254. Introduction. The geometric facts proved in Book I were largely concerned with congruent triangles and parallel lines. In Book II the circle was the fundamental figure, and the truths proved dealt with the measurement of angles in a circle and with lines connected with it. In both books the methods of proof used congruent triangles, pairs of equal lines, and pairs of equal angles. The fundamental idea in Book III is very different, as it concerns not congruent triangles but similar ones, that is, triangles which are of the same shape but of different size. This means that the corresponding angles are equal but that the corresponding sides are not necessarily equal but are proportional. Hence, as in the previous books, pairs of equal angles in triangles play an important part. Instead of dealing continually with pairs of equal lines, however, we must deal with two equal ratios, that is, with groups of *four* lines which are proportional. This makes it necessary at the outset to obtain some clear notions of ratio, proportion, and measurement which are really of an algebraic nature.

255. Ratio. The ratio of one number, a, to a second number, b, is the quotient obtained by dividing the first

by the second, or $\frac{a}{b}$. The ratio of a to b is also written $a : b$, the colon meaning "divided by."

All ratios, therefore, are fractions, and all fractions are ratios.

Thus, $4 : 7$, $\frac{a}{3m}$, $\frac{m-a}{m+a}$, $\frac{5}{6}$ are ratios and fractions.

Any whole number is essentially a ratio, since $6 = \frac{6}{1}$ and $n = \frac{n}{1}$.

256. Measurement. A ratio exists between any two concrete numbers if they have a common unit of measure. The ratio of 4 rd. to 7 yd. is $\frac{22}{7}$, the common unit of measure being the yard. A ratio exists only between magnitudes of the same kind. It goes without saying that magnitudes of different kinds, as time and distance, cannot be measured by means of the same unit.

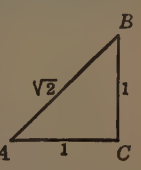
If we say a piece of paper contains 32 sq. in., we are expressing by the number 32 the ratio of the surface of the paper to the surface of a square whose side is 1 in.

Every measurement, then, is the determination of a ratio, either exact or approximate. In measuring two magnitudes which are of the same kind, as, for example, two lines, it is essential that they have a common unit of measure. It will be seen that even this is not always possible.

257. Commensurable magnitudes. Two magnitudes having a common unit of measure are said to be commensurable, and their ratio can be given by two integers.

258. Incommensurable magnitudes. If two magnitudes of the same kind have no common unit of measure, they are said to be incommensurable; that is, the ratio of one to the other cannot be expressed exactly by the quotient of any two whole numbers.

In the adjacent right triangle the shorter sides are each 1 in. long. Hence the hypotenuse and the other two sides are incommensurable. For $\sqrt{2}=1.414213+$. Thus if the unit of measure is .01 in., then the hypotenuse is 141 units long and a little more, but either of the other sides is exactly 100 units long. If the unit of length is .001 inch, the hypotenuse is a little over 1414 units, and the other sides are exactly 1000 units each.



In this case no matter how minute we make the equal divisions of AB, one of them is never contained a whole number of times in AC. This illustrates the fact that AB and AC have no common unit of measure.

It will be seen later that the circumference of a circle (C) and its diameter (D) furnish another example of two magnitudes having no common unit of measure. Here $C \div D = 3.14159+$, a never-ending, nonrepeating decimal. For the present it is sufficient to get the idea that incommensurable numbers are possible.

EXERCISES

Write the following ratios as fractions and, if possible, reduce the fractions to their lowest terms:

1. $48 : 72$.
2. $2\frac{3}{4} : 4\frac{1}{8}$.
3. 250 lb. : 1 T.
4. 32 in. : 8 yd.
5. $(a+5) : (a^2-25)$.
6. $(x^2-4) : (x^2-5x+6)$.
7. $\left(9-\frac{1}{n^2}\right) : \left(3-\frac{1}{n}\right)$.
8. $\frac{1}{x^2-16} : \frac{1}{x^2-4x-32}$.
9. $a^3+b^3 : a+b$.
10. $a-2 : a^3-8$.
11. $\left(1-\frac{a}{m}\right) : \left(1-\frac{m}{a}\right)$.

Find the ratio of x to y if

12. $3x = 12y$. 15. $ax = by$. 18. $\frac{x-y}{x+y} = \frac{2}{5}$.

HINT. Divide by $3y$.

13. $x = 6y$. 16. $mx = (a+c)y$. 19. $\frac{2x-y}{3x+2y} = m$.

14. $9y = 6x$. 17. $ax + by = cx$.

20. Separate 75 into two parts which are in the ratio of 1 : 4.

HINT. Let x = one part. Then $4x$ = the other, and $x + 4x = 75$, etc.

21. Separate 156 into two parts in the ratio of 5 : 7.

22. Separate 980 into three parts in the ratio of 2 : 3 : 9.

23. The value of a ratio is $\frac{3}{4}$, and the second term of the ratio is 15 greater than the first. Find the ratio.

24. The perimeter of a triangle is 30 in., and the sides are in the ratio of 4 : 2 : 3. Find the length of each side.

25. The perimeter of a rectangle is 160 ft., and the sides are in the ratio of 3 : 5. Find the length of each side.

26. The angles of a triangle are in the ratio of 2 : 3 : 4. Find the number of degrees in each angle.

27. Two supplementary angles are in the ratio of 5 : 7. Find each.

28. Find the number of degrees in each angle of a quadrilateral if they are in the ratio of 4 : 5 : 6 : 9.

29. The perimeter of a pentagon is 480 ft., and the sides are in the ratio of 2 : 3 : 4 : 6 : 9. Find the length of each side.

30. The angles of a pentagon are in the ratio of 2 : 3 : 4 : 5 : 6. Find each angle.

259. Proportion. Four numbers, a, b, c, and d, are in *proportion* if the ratio of the first pair equals the ratio of the second pair.

This proportion is written in either of the forms $a : b = c : d$ or $\frac{a}{b} = \frac{c}{d}$.

The first form is much used, as it saves space in printing. The second form is preferable, however, since it is the usual algebraic way of writing an equation. To this form of equation the axioms and rules of operation learned in the study of algebra can be readily applied.

260. Terms of a proportion. The first and fourth terms of a proportion are called the *extremes* and the second and third terms the *means*. Since a proportion contains two ratios, the first and third terms may be called the numerators and the second and fourth the denominators.

261. Fourth proportional. A fourth proportional to three numbers a, b, and c is the number x if $a : b = c : x$.

262. Mean proportional. A mean proportional to two numbers a and b is the number x if $a : x = x : b$.

263. Fundamental forms of a proportion. In the geometric figures to be treated there will frequently be one or more groups of lines which are proportional. The terms of these proportions will be the numeric measures of the lines referred to. The proportion $a : b = c : d$ can be written $b : a = d : c$ and also in many other forms. These various forms are necessary in order to prove the geometric properties of the figures studied. Hence, as a foundation for the geometric work which is to follow, the various forms which a proportion may take must be stated and proved to be correct. Each proof of this kind is equivalent to proving an algebraic theorem. Proofs of the more important forms that a proportion may take follow.

1. *Product of means and extremes.* In any proportion the product of the means equals the product of the extremes.

Proof. If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

Multiply both members of the proportion by bd. Axiom 5

2. *Two equal products.* If the product of two numbers equals the product of two others, either pair may be made the means of a proportion of which the other two are the extremes.

Proof. If $ad = bc$, then $\frac{a}{b} = \frac{c}{d}$.

Divide both members of the given equation by bd. Axiom 6

3. *Alternation.* If four numbers are in proportion, they are in proportion by *alternation*; that is, the first is to the third as the second is to the fourth.

Proof. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$.

Multiply both members of the proportion by $\frac{b}{c}$. Axiom 5

4. *Inversion.* If four numbers are in proportion they are in proportion by *inversion*; that is, the second is to the first as the fourth is to the third.

Proof. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$.

Divide the identity $1 = 1$ by the left and right members of the given equation. Axiom 6

5. *Addition.* If four numbers are in proportion, they are in proportion by *addition*; that is, the sum of the first two is to the second as the sum of the last two is to the fourth.

Proof. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{b} = \frac{c+d}{d}$.

Add 1 to each member of the given equation and combine terms. Axiom 3

6. *Subtraction.* If four numbers are in proportion, they are in proportion by *subtraction*; that is, the difference of the first two is to the second as the difference of the last two is to the fourth.

Proof. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a-b}{b} = \frac{c-d}{d}$.

Subtract 1 from each member of the given equation and then combine terms. Axiom 4

7. *General addition theorem.* In a series of equal ratios the *sum of the numerators* is to the *sum of the denominators* as any numerator is to its denominator.

Proof. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = r$, then $\frac{a+c+e}{b+d+f} = r = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}$.

From the given equations, $a = br$, $c = dr$, $e = fr$. Axiom 5

$\therefore a + c + e = r(b + d + f)$. Axiom 3

Hence $\frac{a+c+e}{b+d+f} = r = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}$.

Each of these seven forms of a proportion may be regarded as an algebraic theorem stated verbally and proved.

EXERCISES

Find the value of x in the following:

1. $8 : 10 = 12 : x$.
2. $(x + 3) : 5 = (x - 1) : 10$.
3. $\frac{8}{4} = \frac{10}{2x - 5}$.
4. Find a fourth proportional to 3, 8, and 6.
5. Find a mean proportional to 12 and 3.
6. State whether the following are true proportions:

(*a*) $\frac{2}{5} = \frac{4}{20}$; (*b*) $\frac{\frac{3}{4}}{\frac{4}{3}} = \frac{2}{2\frac{1}{8}}$; (*c*) $\frac{6}{3\frac{1}{2}} = \frac{8}{4\frac{1}{2}}$.

7. Since $3 \times 6 = 9 \times 2$, write four proportions involving these four numbers.

8. If $uv = wx$, write four proportions involving these letters.

Write the following as proportions in which x is the fourth term:

9. $ax = mn$. 10. $x = \frac{c^2}{n}$. 11. $x = \frac{bc}{a}$. 12. $x = mn$. 13. $x = \frac{a}{bc}$.

Write the following as proportions:

14. $ab = x^2 - 9$.

15. $xy = a^2 + 2ab + b^2$.

16. $x^2 - m^2 = a^2 - b^2$.

17. $x^2 - 5x + 6 = x^2 - 16$.

State from what proportions the following are obtained:

18. $x = \sqrt{m \cdot n}$.

19. $x = \sqrt{m^2 - n^2}$.

20. Write the proportion $\frac{r}{s} = \frac{v}{w}$

(*a*) by alternation, and the resulting proportion by inversion;

(*b*) by inversion, and the resulting proportion by addition;

(*c*) by alternation, and the resulting proportion by subtraction;

(*d*) by addition, and the resulting proportion by alternation.

Given $a : b = c : d$, verify the following proportions:

21. $ma : mb = c : d$. 22. $a : b = cd : d^2$. 23. $a^2 : b^2 = c^2 : d^2$.

Given $a : b = b : c$, show whether the following proportions are true:

24. $b = \sqrt{ac}$.

25. $a : c^2 = b^2 : c^2$.

26. $\frac{ac - 2}{b + 2} = \frac{b - 2}{2}$.

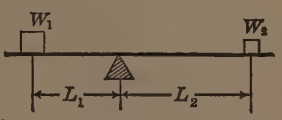
27. $\frac{a - b}{b - c} = \frac{a + b}{b + c}$.

28. $\frac{a - b}{a + b} = \frac{b - c}{b + c}$.

29. $(b - \sqrt{ac})(b + \sqrt{ac}) = 0$.

30. Prove that a mean proportional between two numbers equals the square root of their product.

31. It is demonstrated in physics that if the weight W_1 balances the weight W_2 then $W_1L_1 = W_2L_2$. This is the usual statement of the law of the lever. State this law as a proportion.



32. Two boys, weighing 80 and 100 lb. respectively, balance on a teeter board 12 ft. long. How far is each from the bolt on which the board is balanced?

33. A certain beam balance gave incorrect weights because the arms which should have been equal were slightly unequal. An object placed in the right pan was balanced by a weight w_1 in the left. When the object was placed in the left pan it was balanced by a different weight w_2 in the right pan. Show that the true weight is $x = \sqrt{w_1 w_2}$.

34. Ten men can do a certain piece of work in 12 days. In how many days can eight men do the same work?

HINT. It often happens that two quantities are so related that one *increases* as the other *decreases*; that is, they are *inversely* proportional. Here, for example, as the number of men working is increased the number of days required to do the work is decreased, and vice versa. Hence the proportion which exists may be stated in either of two ways: $8 : 10 = 12 : x$ or $10 : 8 = \frac{1}{12} : \frac{1}{x}$.

35. It requires 24 yd. of carpet 1 yd. wide to cover a certain floor. How many yards $\frac{3}{4}$ yd. wide will be required to cover the same floor?

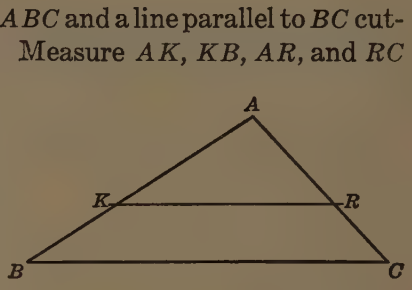
36. In a camp 48 men have provisions to last for 36 days. How many men must leave so that the provisions will last those who remain 54 days?

Introduction to Theorem 1

EXERCISES

In the following exercises it is imperative to use a sharp-pointed pencil. In measuring the lines the length in inches and tenths of an inch should be noted, and the decimal fraction of a tenth over should be estimated as carefully as possible. This will give the length to *hundredths* of an inch.

1. Draw a large triangle ABC and a line parallel to BC cutting AB in K and AC in R. Measure AK, KB, AR, and RC carefully. Then see if the lengths of these four lines in the order named form a true proportion. Does the product of the means equal the product of the extremes very nearly?



2. Repeat Exercise 1 with a different figure.

3. Repeat Exercise 1 with still another figure.

4. Complete the following in accord with the results of Exercises 1 to 3:

If a line is parallel to one side of a triangle, it divides the other two sides into four ____ parts.

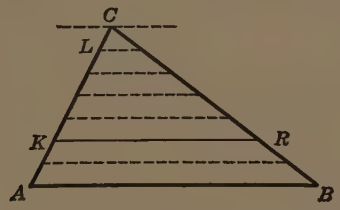
Measure the lines in one of your triangles which is lettered like the figure in Exercise 1, and determine whether the following proportions are true:

5. $\frac{AK}{KB}=\frac{AR}{RC}$. 6. $\frac{AB}{AK}=\frac{AC}{AR}$. 7. $\frac{KR}{BC}=\frac{AK}{AB}$. 8. $\frac{KR}{BC}=\frac{BK}{RC}$.

9. Complete the following statement: If a line parallel to one side of a triangle cuts the other two sides, the corresponding sides of the triangles formed are ____.

Theorem 1

264. *If a line is parallel to one side of a triangle, it divides the other two sides proportionally.*



Given $\triangle ABC$, in which KR, $\parallel$ to AB, intersects AC and BC at K and R, giving the segments CK, AK, CR, and BR.

To prove that $CK:AK = CR:BR$.

Case I. When AK and CK are commensurable

Method. Apply to AK and CK a common unit of measure CL. Suppose it is contained h times in CK and m times in AK. Through C and the points of division draw parallels to AB cutting BC. Then prove that $CK:AK = CR:BR$.

Proof

1. Let the unit of measure CL be contained h times in CK and m times in AK. Through the points of division draw lines $\parallel$ to AB.	1. § 51, Postulate 10.
2. Then CR will be divided into h equal parts and BR into m equal parts.	2. § 144.
3. $\frac{CK}{AK} = \frac{h}{m}$.	3. By construction.
4. Also, $\frac{CR}{BR} = \frac{h}{m}$.	4. Steps 1 and 2.
5. $\therefore CK:AK = CR:BR$.	5. § 50, Axiom 1.

Case II. When AK and CK have no common unit of measure.

The truth of the theorem in this case *will be assumed without proof.* It can be proved by what is called the method of limits.

265. *Corollary. If a line parallel to one side of a triangle cuts the other two sides, either side is to one of its segments as the other side is to the corresponding segment.*

Given the figure as in § 264.

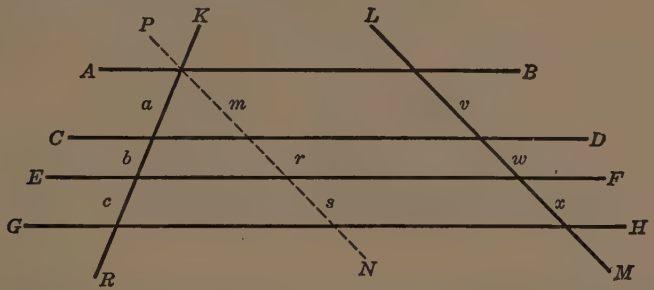
To prove that $\frac{AC}{AK} = \frac{BC}{BR}$ **and that** $\frac{AC}{CK} = \frac{BC}{CR}$.

Proof

1. $\frac{CK}{AK} = \frac{CR}{BR}$.	1. § 264.
2. $\frac{CK + AK}{AK} = \frac{CR + BR}{BR}$.	2. § 263, 5. If four numbers are in proportion, they are in proportion by addition; that is, the sum of the first two is to the second as the sum of the last two is to the fourth.
3. $\therefore \frac{AC}{AK} = \frac{BC}{BR}$.	3. For $CK + AK = AC$ and $CR + BR = BC$.
4. Also, $\frac{AK}{CK} = \frac{BR}{CR}$.	4. From step 1 and § 263, 4. If four numbers are in proportion, they are in proportion by inversion; that is, the second is to the first as the fourth is to the third.
5. $\frac{AK + CK}{CK} = \frac{BR + CR}{CR}$.	5. § 263, 5.
6. $\frac{AC}{CK} = \frac{BC}{CR}$.	6. For $AK + CK = AC$ and $BR + CR = BC$.

Theorem 2

266. *Parallel lines intercept proportional segments on any two transversals.*



Given the ∥ lines AB, CD, EF, and GH cut by KR and LM.

To prove that $\frac{a}{v} = \frac{b}{w} = \frac{c}{x}$.

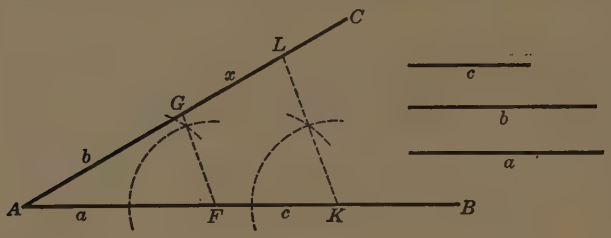
Method. Draw PN through the intersection of AB and KR and parallel to LM, and prove that the corresponding segments of PN and KR are proportional. Then complete the proof by showing that the corresponding segments of PN and LM are equal.

Proof

1. Draw $PN \parallel$ to LM.	1. § 99.
2. $\frac{a+b}{a} = \frac{m+r}{m}$.	2. § 265.
3. $\frac{a+b}{m+r} = \frac{a}{m} = \frac{b}{r}$.	3. § 263, 3, and § 265.
4. $\frac{a+b}{m+r} = \frac{c}{s}$.	4. § 264.
5. $\therefore \frac{a}{m} = \frac{b}{r} = \frac{c}{s}$.	5. Steps 3 and 4.
6. $m = v$, $r = w$, and $s = x$.	6. § 126.
7. $\therefore \frac{a}{v} = \frac{b}{w} = \frac{c}{x}$.	7. Steps 5 and 6, and § 50, 7.

Construction 1

267. *Construct a line segment which is a fourth proportional to three given line segments.*



Given lines *a*, *b*, and *c*.

Required to construct a line x so that a : b = c : x.

Method. Draw AB and AC, making $\angle A$ about 30°. On AB lay off $AF = a$ and $FK = c$, and on AC lay off $AG = b$. Draw FG and through K construct $KL \parallel$ to FG. Then GL is the required fourth proportional.

Proof

1. FG is $\parallel$ to KL.	1. By construction, § 99.
2. $\therefore a : b = c : x$.	2. § 264.

EXERCISES

1. Construct x if $x = \frac{ab}{c}$; a, b, and c being lines of a given length.

HINT. Write $x = \frac{ab}{c}$ as a proportion.

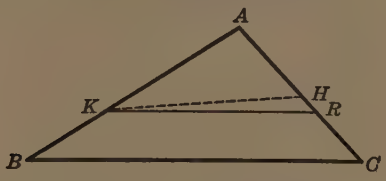
2. Construct x if $a^2 = bx$.

3. Divide a given line into three parts which are proportional to three given lines a, b, and c.

HINT. How can § 266 be used as § 264 was in Construction 1 above?

Theorem 3

268. *If a line divides two sides of a triangle proportionally, it is parallel to the third side.*



Given $\triangle ABC$, in which KR divides AB and AC so that $\frac{AK}{KB}=\frac{AR}{RC}$.

To prove that KR is $\parallel$ to BC.

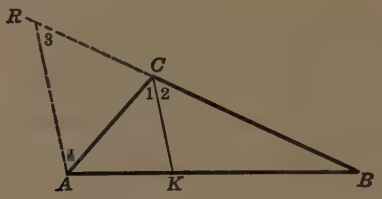
Method. Indirect. Suppose that KR is not $\parallel$ to BC and draw $KH \parallel$ to BC. Then prove that both KR and KH will divide AC into segments proportional to AK and KB. Hence KR and KH must coincide. Then KR is $\parallel$ to BC.

Proof

1. Draw $KH \parallel$ to BC.	1. § 99.
2. Then $\frac{AK}{KB}=\frac{AH}{HC}$.	2. § 264.
3. $\frac{AK}{KB}=\frac{AR}{RC}$.	3. By hypothesis.
4. $\frac{AR}{RC}=\frac{AH}{HC}$.	4. § 50, Axiom 1.
5. $\frac{AR+RC}{RC}=\frac{AH+HC}{HC}$.	5. § 263, 5.
6. $\frac{AC}{RC}=\frac{AC}{HC}$.	6. For $AR + RC = AC$ and $AH + HC = AC$.
7. $RC = HC$.	7. From step 6.
8. R coincides with H.	8. From step 7.
9. $\therefore$ KR is $\parallel$ to BC.	9. § 82.

Theorem 4

269. *The bisector of an angle of a triangle divides the side opposite into segments proportional to the sides adjacent to them.*



Given $\triangle ABC$, **with** CK **bisecting** $\angle ACB$ **and meeting** AB **at** K.

To prove that $AK : KB = AC : BC$.

Method. Draw through A a line parallel to the bisector meeting BC produced in R, and prove that CK divides the sides of $\triangle ABR$ proportionally. Then show that $AC = RC$, and substitute AC for RC in the proportion just obtained. This will be the required result.

Proof

1. Through A draw a line $\parallel$ to KC intersecting BC produced at R.	1. § 99.
2. Then $BK : KA = BC : CR$.	2. § 264.
3. $\angle 1 = \angle 2$.	3. By hypothesis.
4. $\angle 1 = \angle 4$.	4. § 91.
5. $\angle 2 = \angle 3$.	5. § 92.
6. $\therefore \angle 4 = \angle 3$.	6. § 50, Axiom 1.
7. $\therefore CR = CA$.	7. § 107.
8. $\therefore BK : KA = BC : CA$.	8. Steps 2 and 7 and § 50, Axiom 7.

QUERY. Is Theorem 4 true if $AC = BC$?

EXERCISES

1. In the figure of § 264, if $AK = 10$ in., $KC = 18$ in., and $BC = 56$ in., find BR and CR.

2. If, in the figure of § 264, $AC = 20$, $BC = 16$, and $BR = 4$, find the other segments of AC and BC.

3. If, in the figure of § 266, $a = 10$, $b = 12$, $c = 8$, and $v + w + x = 45$, find v, w, and x.

4. In the figure of § 268, if $AK = 20''$, $KB = 8''$, $AR = 16''$, and $RC = 6''$, will KR be parallel to BC?

5. In the figure of § 268, $AK = 40'$, $KB = 24'$, and $AC = 48'$. How long must AR be if KR is $\parallel$ to BC?

6. The sides of a triangle are 10, 15, and 20 respectively. Find the segments of the side 20 made by the bisector of the angle opposite.

7. The perimeter of a triangle is 60 ft., and the bisector of one angle divides the side opposite into segments of 8 ft. and 12 ft. Find the other two sides.

8. Is the converse of Theorem 4 true? If so, prove.

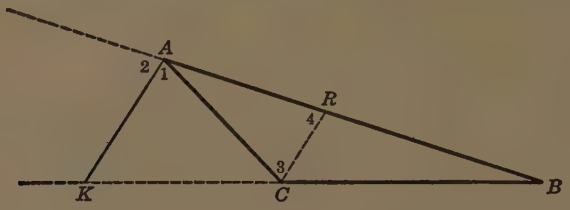
270. Segments of a line made by a point. If, in the adjacent figure, AB is a line of given length and K is a point on AB or AB produced, KA and KB are called the segments of the line AB.

When the point of division is on the line, the line is divided *internally*; when the point of division is on the line produced in either direction, the line is divided *externally*.

Note that in either case the segments are measured *from the point of division to the extremities of the line*, and that when the point is on the line produced, one segment is longer than the line itself.

Theorem 5

271. *If a triangle is not isosceles, the bisector of an exterior angle at any vertex divides the side opposite externally into segments proportional to the adjacent sides.*



Given $\triangle ABC$, in which AK, the bisector of the exterior angle at A, meets the side BC produced at K.

To prove that $KB : KC = AB : AC$.

Method. Draw through C a parallel to the bisector, meeting AB in R, and prove that CR divides two sides of $\triangle KBA$ proportionally. Then prove that $AC = AR$, and substitute AC for AR in the proportion obtained, and this will be the required result.

Proof

1. Draw through C a line $\parallel$ to AK meeting AB at R.	1. § 99.
2. Then $KB : KC = AB : AR$.	2. § 265.
3. $\angle 1 = \angle 2$.	3. By hypothesis.
4. $\angle 2 = \angle 4$.	4. § 92.
5. $\angle 1 = \angle 3$.	5. § 91.
6. $\angle 4 = \angle 3$.	6. § 50, Axiom 1.
7. $AC = AR$.	7. § 107.
8. $\therefore KB : KC = AB : AC$.	8. Steps 2 and 7 and § 50, Axiom 7.

EXERCISES

1. The sides of a triangle are 8 ft., 12 ft., and 15 ft. Find the segments of the side measuring 15 ft. made by the bisector of the exterior angle opposite.

2. The perimeter of a triangle is 45 ft., and one side is 20 ft. The segments of this side made by the bisector of the exterior angle opposite are in the ratio of 2 : 3. Find the other two sides of the triangle.

SIMILAR POLYGONS

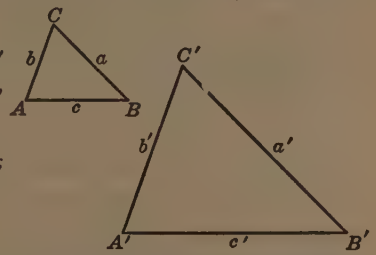
272. Similar polygons. Polygons which have the same shapes are similar. Conversely,

If two polygons are similar (symbol ~), the corresponding angles are equal and the corresponding sides are proportional.

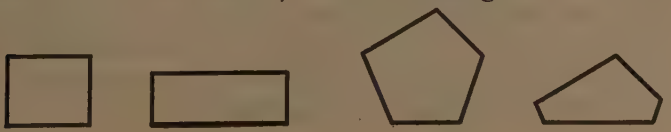
273. Corresponding angles and sides. In similar (or congruent) polygons the respectively equal angles are *corresponding* angles, and two sides which are in like position with respect to the equal angles are called *corresponding* or homologous sides; that is, corresponding sides lie opposite equal angles.

Thus, if $\triangle ABC \sim \triangle A'B'C'$ and $\angle A = \angle A'$, $\angle B = \angle B'$, and $\angle C = \angle C'$, then

$$\frac{a, \text{ opposite } \angle A}{a', \text{ opposite } \angle A'} = \frac{b, \text{ opposite } \angle B}{b', \text{ opposite } \angle B'} = \frac{c, \text{ opposite } \angle C}{c', \text{ opposite } \angle C'}.$$

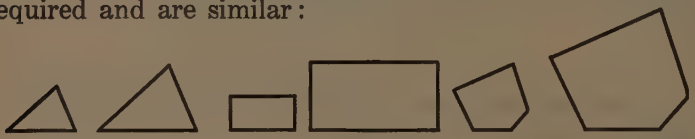


Two polygons may have their corresponding angles equal and not be similar, as the following:



Two polygons may have their sides proportional and not be similar, as are the following pairs:

The following pairs of polygons have both the properties required and are similar:



QUERY 1. Must similar polygons have the same number of sides? of angles?

QUERY 2. Are all squares similar? all rectangles? all equilateral triangles?

QUERY 3. State whether the corresponding sides of the following polygons are proportional: (*a*) two squares; (*b*) two rectangles; (*c*) two parallelograms; (*d*) two regular polygons; (*e*) two regular polygons having the same number of sides.

QUERY 4. Are equal triangles similar?

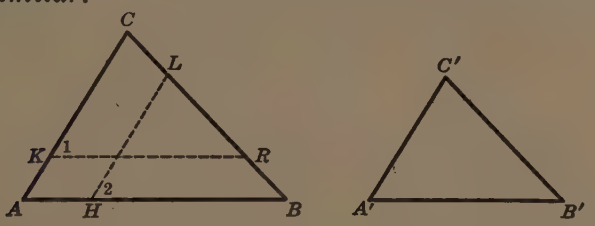
QUERY 5. Are equal polygons similar?

QUERY 6. In the three rectangles of the adjacent figure, $b : a = a : x$. Are any two of the three rectangles similar? Which? Prove.

274. Uses of similar figures. The usefulness of maps lies in two facts: (*a*) they are the same shape as the actual region pictured, and (*b*) they are drawn to scale. The first insures that the direction of one point from another point on the map is the same as in the actual case. The second makes corresponding distances in map and country proportional. In like manner the plans of the architect and the engineer and the blue prints of the machinist are based on the properties of similar figures.

Theorem 6

275. *If two triangles are mutually equiangular, they are similar.*



Given $\triangle ABC$ and $A'B'C'$, in which $\angle A = \angle A'$, $\angle B = \angle B'$, and $\angle C = \angle C'$.

To prove that $\triangle ABC \sim \triangle A'B'C'$.

Method. Superpose one triangle on the other so that equal angles coincide. Then prove that the third side of one is parallel to the third side of the other and that two pairs of sides are proportional. Repeat these steps for another angle, and show that the corresponding sides of the two triangles are proportional.

Proof

1. Place $\triangle A'B'C'$ on $\triangle ABC$ so that C' falls on C, $C'A'$ on CA, $C'B'$ on CB, and $A'B'$ in the position KR.	1. By hypothesis, $\angle C = \angle C'$; § 51, Postulate 7.
2. $\angle 1 = \angle A' = \angle A$.	2. By hypothesis.
3. Hence KR is $\parallel$ to AB.	3. § 97.
4. $\frac{CK}{CA} = \frac{CR}{CB}$.	4. § 265.
5. $\frac{C'A'}{CA} = \frac{C'B'}{CB}$.	5. Steps 1 and 4; § 50, Axiom 7.
6. Place $\triangle A'B'C'$ on $\triangle ABC$ so that B' falls on B, $B'C'$ on BC, $A'B'$ on AB, and $A'C'$ in the position HL.	6. By hypothesis, $\angle B = \angle B'$; § 51, Postulate 7.

7. $\angle 2 = \angle A' = \angle A$.	7. By hypothesis.
8. Hence HL is $\parallel$ to AC.	8. § 97.
9. $\frac{BL}{BC} = \frac{BH}{BA}$, or $\frac{B'C'}{BC} = \frac{A'B'}{AB}$.	9. § 265; § 50, Axiom 7.
10. Hence $\frac{C'A'}{CA} = \frac{C'B'}{CB} = \frac{A'B'}{AB}$.	10. Steps 5 and 9.
11. $\therefore \triangle ABC \sim A'B'C'$.	11. By hypothesis, step 10, and § 272.

276. *Corollary 1. If two angles of one triangle are equal respectively to two angles of another, the triangles are similar.*

277. *Corollary 2. If two right triangles have an acute angle of one equal to an acute angle of the other, the triangles are similar.*

278. *Corollary 3. A line cutting two sides of a triangle and parallel to the third side forms a second triangle similar to the first.*

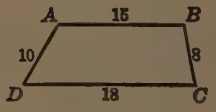
279. *Corollary 4. Two triangles similar to a third triangle are similar to each other.*

EXERCISES

1. The sides of a triangle are $a = 7$, $b = 8$, and $c = 10$. In a similar triangle $b' = 21$; find a' and c'.

2. A flagpole casts a shadow 72 ft. long at the same time that a lamp post 12 ft. high casts a shadow 8 ft. long. Find the height of the flagpole.

3. In the adjacent trapezoid, how far must DA and CB be produced in order to intersect?

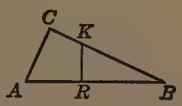


4. The sides of a triangle are $a = 10$ in., $b = 12$ in., and $c = 18$ in. A line 8 in. long parallel to side c terminates in the other two sides. Find the segments into which it divides sides a and b.

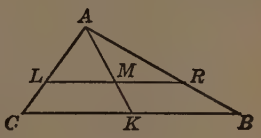
5. Prove that any two equilateral triangles are similar.

6. C is the right angle of $\triangle ABC$. From any point K on BC a perpendicular is drawn to AB, meeting it in R.

Prove $\triangle ABC \sim \triangle BKR$.



7. Prove that the median to the base of a triangle bisects any line parallel to the base and terminating in the other two sides.

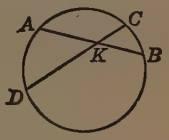


8. In rt. $\triangle ABC$ line CK is $\perp$ to the hypotenuse AB.

Prove $\triangle ABC \sim \triangle ACK \sim \triangle BCK$.

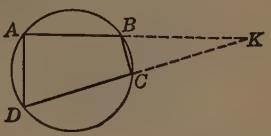
9. Chords AB and CD intersect at K. Draw AC and BD.

Prove $\triangle ACK \sim \triangle BDK$.



10. $ABCD$ is an inscribed quadrilateral. AB and DC produced intersect at K.

Prove $\triangle KBC \sim \triangle KAD$.

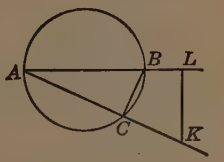


11. $ABCD$ is a trapezoid whose diagonals, AC and BD, intersect at K. (AB and CD are bases.)

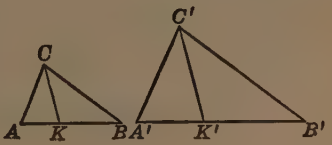
Prove $\triangle AKB \sim \triangle DKC$.

12. AB is the diameter of a circle. From any point K on chord AC produced a perpendicular is drawn to AB, meeting it produced in L.

Prove $\triangle ABC \sim \triangle ALK$.



13. In two similar triangles any two sides are in the same ratio as the bisectors of two equal angles. Prove.



Proportional Lines

280. Method of proving a geometric proportion. It is frequently desirable and even necessary to prove that the lengths of certain lines of a given figure are proportional. Usually such lines are corresponding parts of similar triangles. In order to prove that the proportion is true it is necessary to show that a pair of similar triangles exists which have the four lines of the proportion as two pairs of corresponding sides. The use of Theorem 6 offers an easy method of proving that two triangles are similar. If two triangles are similar, the corresponding sides lie opposite the equal angles. The method then consists of two steps:

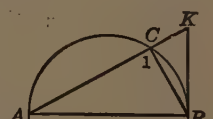
(*a*) Select the proper pair of triangles and prove that they are similar.

(*b*) Show that the four lines of the given proportion are two pairs of corresponding sides of the similar triangles.

EXAMPLE

In the adjacent figure, AB is a diameter and KB is a tangent to the circle.

Prove $\overline{KB}^2 = KC \cdot KA$. (1)



Informal proof. Equation (1) becomes $KA : KB = KB : KC$. This proportion suggests that $\triangle KAB \sim \triangle KBC$. Draw BC. Then $\angle A = \angle KBC$, since each is measured by one half arc BC. Also, $\angle 1 = \angle ABK = 90°$, since each is measured by one half a semicircle. Therefore $\triangle KAB \sim \triangle KBC$, having two angles of one equal respectively to two angles of the other. Therefore in $\triangle KAB$ and KBC,

$$\frac{KA, \text{ opposite } \angle ABK}{KB, \text{ opposite } \angle KCB} = \frac{KB, \text{ opposite } \angle A}{KC, \text{ opposite } \angle KBC}.$$

This proof illustrates three important points:

(*a*) An equation containing two equal products should be written as a proportion.

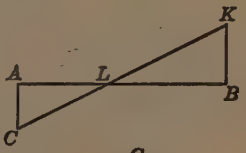
(*b*) The proportion often suggests the triangles which should be proved similar.

(*c*) Each ratio of the proportion to be proved correct is a pair of lines, one in each triangle, lying opposite two equal angles.

EXERCISES ON PROPORTIONAL LINES OF SIMILAR TRIANGLES

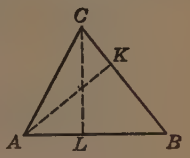
1. In the adjacent figure, $\angle A$ and $\angle B$ are right angles.

Prove $\frac{AC}{BK} = \frac{AL}{BL} = \frac{CL}{KL}$.



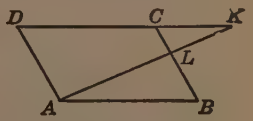
2. In the adjacent figure, AK and CL are altitudes of $\triangle ABC$.

Prove $AB : BK = BC : BL$.



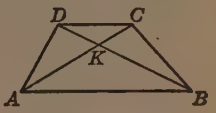
3. $ABCD$ in the adjacent figure is a parallelogram.

Prove $AK : AL = AD : BL$.



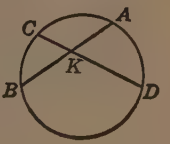
4. The diagonals of trapezoid $ABCD$ intersect in K.

Prove $AK \cdot DK = BK \cdot CK$.



5. Two chords, AB and CD, intersect at K.

Prove $AK \cdot BK = CK \cdot DK$.

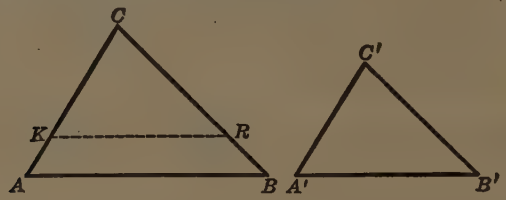


6. Chords AB and CD produced intersect at K.

Prove $KB \cdot KA = KD \cdot KC$.

Theorem 7

281. *If an angle of one triangle equals an angle of another, and the sides including those angles are proportional, the triangles are similar.*



Given $\triangle ABC$ and $\triangle A'B'C'$, in which $\angle C = \angle C'$ and $\frac{CA}{C'A'} = \frac{CB}{C'B'}$.

To prove that $\triangle ABC \sim \triangle A'B'C'$.

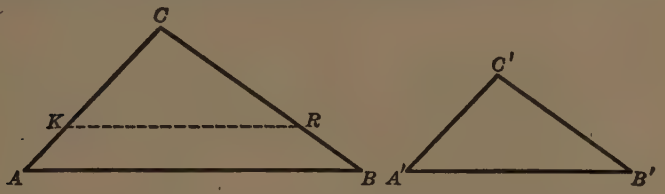
Method. Place $\triangle A'B'C'$ on $\triangle ABC$ so that $\angle C'$ falls on $\angle C$, $C'A'$ on CA, and $C'B'$ on CB. Then prove that $A'B'$ in its new position will be parallel to AB. It then follows that $\triangle A'B'C' \sim \triangle ABC$.

Proof

1. Place $\triangle A'B'C'$ on $\triangle ABC$ so that C' falls on C, $C'A'$ on CA, and $C'B'$ on CB. Then $A'B'$ will fall in the position KR.	1. By hypothesis, $\angle C = \angle C'$; § 51, Postulate 7.
2. $\frac{CA}{C'A'} = \frac{CB}{C'B'}$, or $\frac{CA}{CK} = \frac{CB}{CR}$.	2. By hypothesis.
3. Hence KR is $\parallel$ to AB.	3. § 268.
4. $\angle A = \angle CKR$.	4. § 92.
5. $\therefore \triangle KRC \sim \triangle ABC$.	5. § 276.
6. Or $\triangle ABC \sim \triangle A'B'C'$.	6. $\triangle A'B'C' \cong \triangle KRC$.

Theorem 8

282. *If the sides of two triangles are respectively proportional, the triangles are similar.*



Given $\triangle ABC$ and $A'B'C'$, in which $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}$.

To prove that $\triangle ABC \sim \triangle A'B'C'$.

Method. On CA lay off CK equal to $C'A'$, and on CB lay off CR equal to $C'B'$. Draw KR, and prove that $\triangle CKR$ is similar to $\triangle ABC$ and congruent to $\triangle A'B'C'$. It then follows that $\triangle ABC \sim \triangle A'B'C'$.

Proof

1. Locate K and R, and draw KR.	1. § 51, Postulate 1.
2. $\frac{CA}{C'A'} = \frac{BC}{B'C'} = \frac{AB}{A'B'}$.	2. By hypothesis.
3. But $C'A' = KC$, and also $B'C' = RC$.	3. By construction.
4. Hence $\frac{CA}{KC} = \frac{BC}{RC} = \frac{AB}{A'B'}$.	4. § 50, Axiom 7.
5. Then $\triangle CKR \sim \triangle CAB$.	5. § 281.
6. Hence $\frac{CA}{KC} = \frac{BC}{RC} = \frac{AB}{KR}$.	6. § 272.
7. Then $\frac{AB}{A'B'} = \frac{AB}{KR}$.	7. Steps 4 and 6 and § 50, Axiom 1.
8. $\therefore A'B' = KR$.	8. Step 7.
9. Hence $\triangle A'B'C' \cong \triangle KRC$.	9. § 71.
10. $\therefore \triangle A'B'C' \sim \triangle ABC$.	10. § 275.

EXERCISES ON THEOREMS 7 AND 8

1. In $\triangle ABC$, L is on side AB so that $AL = \frac{2}{3} LB$, and M is on side AC so that $AM = \frac{2}{3} MC$.

Prove $\triangle ALM \sim \triangle ABC$.

2. The diagonals of a quadrilateral $ABCD$ intersect at K so that $DK = \frac{3}{5} KB$ and $KC = 15$. The diagonal $AC = 40$. Is the quadrilateral a trapezoid? Prove.

3. The diagonals of a quadrilateral $ABCD$ intersect at K so that $AK \cdot KD = BK \cdot KC$.

Prove that $ABCD$ is a trapezoid.

4. In two similar triangles any two corresponding sides are in the same ratio as any two corresponding medians. Prove.

HINT. Prove $\triangle ACM \sim \triangle A'C'M'$.

5. The sides of a triangle are $4\frac{1}{4}$, $5\frac{1}{3}$, and $6\frac{1}{2}$. The sides of a second triangle are 78, 51, and 64. Are the triangles similar? Prove.

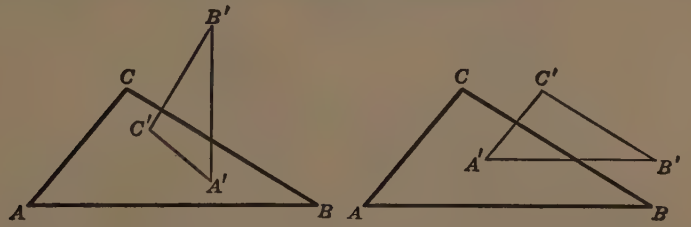
6. The sides of a triangle are 10, 12, and 15. The perimeter of a similar triangle is 185. Find the sides of the second triangle.

Theorems 6 and 8 prove that the triangle is unique among polygons. For if two triangles are mutually equiangular they are similar, and if their sides are proportional they are similar.

Any other two polygons, however, may have their sides proportional without being mutually equiangular, and their angles may be respectively equal without their sides being proportional. Hence one of these conditions alone is not sufficient to make two polygons (triangles excepted) similar. See § 273 for illustrations.

Theorem 9

283. *If two triangles have their sides respectively perpendicular or parallel, they are similar.*



Given $\triangle ABC$ and $A'B'C'$, in which AB, BC, and CA are $\perp$ or $\parallel$ respectively to $A'B'$, $B'C'$, and $C'A'$.

To prove that $\triangle ABC \sim \triangle A'B'C'$.

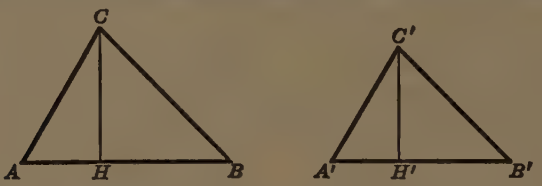
Method. Indirect.

Proof

1. The $\angle$ of the $\triangle$ have their corresponding sides $\perp$ or $\parallel$.	1. By hypothesis.
2. $\therefore$ the corresponding $\angle$ are equal or supplementary.	2. §§ 94 and 106.
3. Either three, two, one, or no pairs of $\angle$ are supplementary.	3. No other assumptions are possible.
4. If two or more of the equations $A + A' = 180°$, $B + B' = 180°$, $C + C' = 180°$ were true, the sum of the $\angle$ of the $\triangle$ would exceed 4 rt. $\angle$, which is impossible.	4. § 100.
5. It is impossible for only one pair to be supplementary.	5. § 103.
6. Hence the $\triangle$ are mutually equiangular.	6. No other possibility exists.
7. $\therefore \triangle ABC \sim \triangle A'B'C'$.	7. § 275.

Theorem 10

284. *If two triangles are similar, corresponding altitudes are proportional to any two corresponding sides.*



Given $\triangle ABC$ similar to $\triangle A'B'C'$, with CH and $C'H'$ corresponding altitudes.

To prove that $\frac{CH}{C'H'} = \frac{AC}{A'C'} = \frac{AB}{A'B'} = \frac{BC}{B'C'}$.

Method. In the right triangles formed, prove that the altitudes are proportional to a pair of corresponding sides of the given triangles. Hence they are proportional to the other two pairs of corresponding sides of the original triangles.

Proof

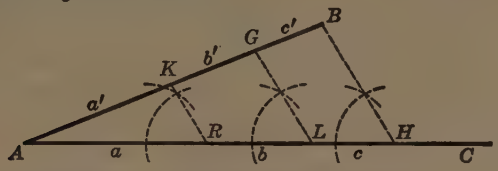
1. $\angle A = \angle A'$.	1. By hypothesis.
2. $\triangle ACH \sim \triangle A'C'H'$.	2. § 277.
3. $\therefore \frac{CH}{C'H'} = \frac{AC}{A'C'}$.	3. § 272.
4. $\frac{AC}{A'C'} = \frac{AB}{A'B'} = \frac{BC}{B'C'}$.	4. § 272.
5. $\therefore \frac{CH}{C'H'} = \frac{AC}{A'C'} = \frac{AB}{A'B'} = \frac{BC}{B'C'}$.	5. § 50, Axiom 1.

285. Corresponding lines of similar polygons. Theorem 10, Exercise 13, page 251, and Exercise 4, page 256, are special cases of the following more general theorem:

In two similar polygons any two lines similarly drawn are in the same ratio as any two corresponding sides.

Construction 2

286. *Divide a given line into parts proportional to two or more given lines.*



Given the lines a, b, and c and the line AB.

Required to divide AB into parts proportional to a, b, and c.

Method. Draw AC, making $\angle A$ about 30°. From A on AC lay off AR, RL, and LH equal respectively to a, b, and c. Draw BH, and through R and L construct parallels to BH, cutting AB at K and G respectively. Then AK, KG, and GB are the required parts of the line AB.

Proof

1. RK is $\parallel$ to LG and LG is $\parallel$ to HB.	1. By construction.
2. $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.	2. § 266.

CONSTRUCTIONS

1. Construct a triangle similar to a given one and having its perimeter equal to a given line segment.

2. Given a rectangle and a line segment. Construct a rectangle similar to the given one and having its perimeter equal to the given line segment.

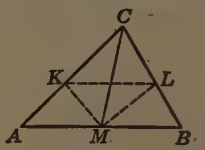
3. Given one side, construct a triangle similar to a given one.

4. Given one altitude, construct a triangle similar to a given one.

287. Connecting ratios. Sometimes it is not possible to prove four lines proportional by the direct use of similar triangles. The use of Theorems 4 and 5 may be required. Occasionally it may be necessary to use a connecting ratio between two other ratios. Thus if it is required to prove $a : b = m : n$ where a, b, m, and n are line segments, it is often possible to show that $a : b = p : q$ and also that $p : q = m : n$. It will then follow that $\frac{a}{b} = \frac{p}{q} = \frac{m}{n}$. Here the connecting ratio $\frac{p}{q}$ establishes the equality of the other two ratios and proves the truth of the given proportion.

EXAMPLE

In $\triangle ABC$ point M is the mid-point of AB. ML bisects $\angle CMB$, and MK bisects $\angle CMA$.



Prove KL is $\parallel$ to AB.

Informal proof. By Theorem 3, KL is $\parallel$ to AB if $AK : KC = BL : LC$.

By Theorem 4, § 269, $AM : MC = AK : KC$ (1)

and $BM : MC = BL : LC$. (2)

By hypothesis, $AM = BM$.

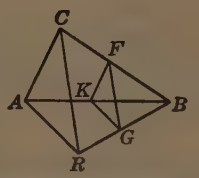
Hence (2) becomes $AM : MC = BL : LC$. (3)

From (1) and (3), $AK : KC = BL : LC$. (4)

Therefore, by Theorem 3, KL is $\parallel$ to AB.

EXERCISES

1. Through K, any point in AB, the common base of $\triangle ABC$ and ABR, lines are drawn parallel to AC and AR meeting BC and BR in points F and G respectively.

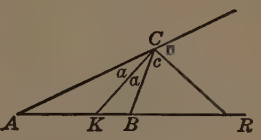


Prove FG is $\parallel$ to CR.

HINT. Prove $BF : BC = BG : BR$ by finding the connecting ratios.

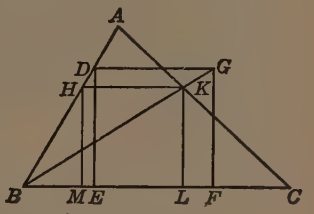
2. The bisectors of the interior and exterior angles at C of $\triangle ABC$ cut the base in K and R respectively.

Prove $KA \cdot RB = RA \cdot KB$.



HINT. This equation may be written $KB : KA = RB : RA$.

3. In the figure, $DEFG$ is a square. The line BG cuts the side AC of the triangle at the point K. Through K two parallels to GD and GF respectively cut AB and BC in H and L respectively. A parallel to FG through H cuts BC in M.

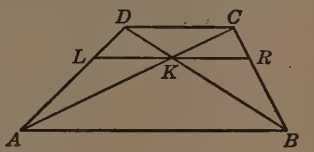


Prove that $HKLM$ is a square.

4. From the preceding exercise devise a method of inscribing in a given triangle either a square or a rectangle similar to a given square or rectangle.

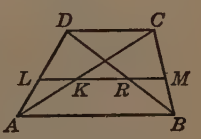
5. A line LR is drawn parallel to one base of a trapezoid through K, the point of intersection of its diagonals, and terminating in the nonparallel sides.

Prove $LK = KR$.



HINT. $LK : DC = AL : AD$, etc.

6. Every line parallel to the bases of a trapezoid and terminating in the nonparallel sides but not containing the intersection of the diagonals is divided into three segments, two of which are equal. Prove.



7. K is the mid-point of BC, one of the parallel sides of trapezoid $ABCD$. The lines AK and DK produced meet DC and AB produced in F and G respectively.

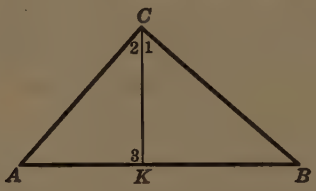
Prove FG is $\parallel$ to AD.

Theorem 11

288. *If the altitude be drawn to the hypotenuse of a right triangle,*

(*a*) *the altitude is a mean proportional between the segments of the hypotenuse;*

(*b*) *either side about the right angle is a mean proportional between the whole hypotenuse and the adjacent segment.*



Given CK, **the altitude on the hypotenuse** AB **of rt.** $\triangle$ ABC.

Method. Prove that the perpendicular forms two right triangles similar (*a*) to each other and (*b*) to the whole triangle. The required proportions follow from the fact that they are composed of two pairs of corresponding sides of similar triangles.

(*a*) ***To prove that*** $AK : CK = CK : KB$.

Proof

1. In $\triangle AKC$ and $\triangle ABC$, $\angle A + \angle 2 = \angle A + \angle B$.	1. § 104.
2. $\therefore \angle 2 = \angle B$.	2. § 50, Axiom 4.
3. $\therefore \triangle AKC \sim \triangle BKC$.	3. § 277.
4. $\frac{AK, \text{ opposite } \angle 2}{CK, \text{ opposite } \angle A} = \frac{CK, \text{ opposite } \angle B}{KB, \text{ opposite } \angle 1}$.	4. § 272.

(b) To prove that $AB:AC = AC:AK$ and $AB:BC = BC:BK$.

Proof

5. Since $\angle A$ is in both $\triangle$, $\triangle ABC \sim \triangle AKC$.	5. § 277.
6. $\angle 3 = \angle ACB$ and $\angle B = 2$.	6. § 272.
7. $\frac{AB, \text{ opposite } \angle ACB}{AC, \text{ opposite } \angle 3} = \frac{AC, \text{ opposite } \angle B}{AK, \text{ opposite } \angle 2}$.	7. § 272.

In a similar manner, prove that $\triangle ABC \sim \triangle BKC$, and hence $AB:BC = BC:BK$.

EXERCISES

Exercises 1–6 refer to the figure of Theorem 11.

1. If AK is 4 and AB is 20, find the length of CK, AC, and BC.

2. If AC is 6 and AB is 18, find the length of AK, CK, and BC.

3. If AB is 36 and CA is 12, find the length of AK and BC.

4. If AC is 12 and BK is 18, find the length of AK, AB, and BC.

5. If AC is 10 and BK is 8, find AB.

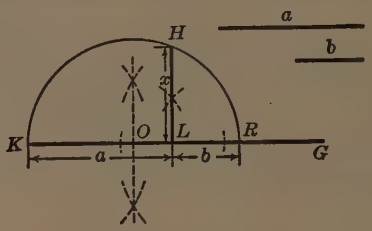
HINT. $\overline{AC}^2 = AB \times AK$. Let $AK = x$. Then $x(x+8) = 100$, and $x^2 + 8x + 16 = 116$; whence $x + 4 = \pm\sqrt{116} = \pm 10.77$. Etc.

6. If AK is 12 and BC is 20, find AB.

7. From any point R on a circle, RK is drawn perpendicular to the diameter AB. Then KH is drawn perpendicular to the radius RL. Show that the proportion $AL:KR = KR:RH$ is true.

Construction 3

289. *Construct a mean proportional between two given line segments.*



Given the lines a and b.

Required to construct a line x so that $a : x = x : b$.

Method. Draw the indefinite straight line KG and lay off KL equal to a and LR equal to b. Bisect KR at O. With O as a center and OK as a radius, construct the semicircle KR. Construct a perpendicular to KR at L, cutting the semicircle at H. HL is the required line.

Proof

1. Consider $\angle KHR$ (not shown in the figure).
2. $\angle KHR = 90°$. — 2. § 230.
3. $\therefore KL : HL = HL : LR$ or $a : x = x : b$. — 3. § 288 (a).

EXERCISES

1. Construct x if $x = \sqrt{ab}$, a and b being lines of given length.

2. Construct a line $\sqrt{6}$ in. long.

HINT. Let x be the line. Then $x^2 = 6$. Therefore $2 : x = x : 3$.

3. Construct a line $a\sqrt{3}$ in. long, a being a line of given length.

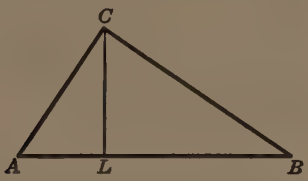
HINT. Let x be $a\sqrt{3}$. Then $x^2 = 3a^2 = a(3a)$, and $a : x = x : 3a$.

4. Construct x if x is $a\sqrt{5}$, a being a given line.

5. Construct x if x is $\frac{a}{2}\sqrt{3}$, a being a given line.

Theorem 12 (*The Pythagorean Theorem*)

290. *In any right triangle the square of the hypotenuse equals the sum of the squares of the other two sides.*



Given rt. $\triangle ABC$, in which AB is the hypotenuse.

To prove that $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

Method. Draw the altitude on the hypotenuse. Use (*b*) of Theorem 11 twice. Then add the two equations thus obtained and simplify the result.

Proof

1. Draw $CL \perp$ to AB.	1. § 76.
2. Then $\overline{AC}^2 = AB \cdot AL$ and $\overline{BC}^2 = AB \cdot BL$.	2. § 288 (*b*).
3. $\overline{AC}^2 + \overline{BC}^2 = AB(AL + BL)$ $= AB \cdot AB = \overline{AB}^2$.	3. § 50, Axiom 3.

Theorem 12 is called the Pythagorean Theorem, after Pythagoras (582–500 B.C.), one of the ablest of the early Greek mathematicians. It seems certain that he was the first geometer to prove this theorem.

EXERCISES

In the following exercises use the figure of Theorem 12:

1. If $AC = 7$ and $BC = 24$, show that $AB = 25$.
2. If $AC = 20$ and $AB = 29$, find BC.
3. If $CL = 15$ and $AL = 8$, find AC, AB, and BC.
4. If $CL = 18$ and $BL = 24$, find BC, AB, and AC.

5. Find the diagonal of a rectangle whose sides are 133 and 156.

6. The diagonal of a rectangle is 277 and one side is 115. Find the other side.

7. AB and AC are equal sides of the isosceles triangle ABC, AK is the mid-perpendicular of BC, and KR is perpendicular to AC, meeting it in R.

Prove $\overline{AK}^2 = AB \cdot AR$.

8. In the figure of Exercise 7 show that $\overline{BC}^2 = 4\,CR \cdot AB$.

9. CK is an altitude of the $\triangle ABC$.

Prove $\overline{AC}^2 + \overline{BK}^2 = \overline{AK}^2 + \overline{BC}^2$.

291. Right triangles in which one angle is 45° or 30°. An important application of Theorem 12 is its use in connection with right triangles in which one acute angle is 45° or in which one is 30°. The two problems which arise in practice when one angle is 45° are illustrated in Figs. 1 and 2, while the two problems involving 30° (or 60°) are

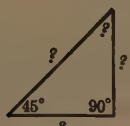


Fig. 1

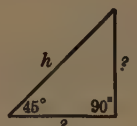


Fig. 2

Fig. 3

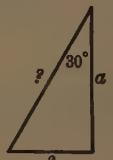


Fig. 4

illustrated in Figs. 3 and 4. Note how the relations proved in Theorems 18 and 19 of Book I are used in 3 and 4:

1. Given one angle of 45° and one side s about the right angle, to find the other acute angle and the other sides.

2. Given one angle of 45° and the hypotenuse h, to find the other acute angle and the other two sides.

3. Given one angle of 30° and the hypotenuse s, to find the other acute angle and the two sides.

4. Given one angle of 30° and one side a about the right angle, to find the other acute angle and the other sides.

The relation of Fig. 4 on page 266 to an equilateral triangle should be noted.

In general, if we know that one angle of any right triangle is 30°, 45°, or 60°, and know also any one side, we can compute the other angle and the other sides.

EXERCISES

1. If the acute angles of a right triangle are each 45° and one side about the right angle is m, show that the hypotenuse is $m\sqrt{2}$.

2. If the acute angles of a right triangle are each 45° and the hypotenuse is s, show that each side is $\frac{s}{2}\sqrt{2}$.

3. If one acute angle of a right triangle is 30° and the hypotenuse is m, show that the longer of the other two sides is $\frac{m}{2}\sqrt{3}$.

4. If one acute angle of a right triangle is 30° and the longer side about the right angle is a, show that the hypotenuse is $\frac{2a}{3}\sqrt{3}$.

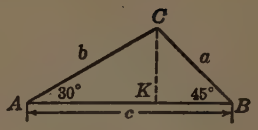
In Exercises 5–9, inclusive, give answers in simplest algebraic form, merely indicating the roots involved:

5. The hypotenuse of an isosceles right triangle is 12. Find the other sides.

6. The side of an equilateral triangle is 22. Find the altitude.

7. The altitude of an equilateral triangle is 15. Find the side.

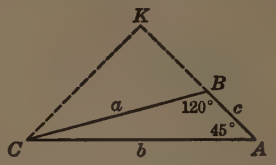
8. In $\triangle ABC$, A is 30°, B is 45°, and b is 20. Find a and c.



9. In $\triangle ABC$, A is 30°, b is 12, and c is 16. Find a.

10. In $\triangle ABC$, A is 45°, B is 120°, and b is 100. Show that c is 29.88 + and a is 81.64+.

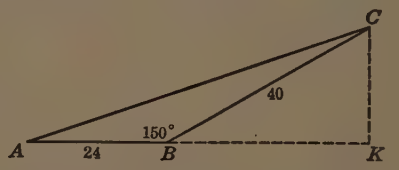
HINT. Draw a perpendicular from C to AB produced.



11. In $\triangle ABC$, A is 45°, b is 40, and c is 80. Show that a is 58.95+.

12. In $\triangle ABC$, A is 30°, B is 135°, and b is 20. Show that c is 7.320+ and a is 14.142+.

13. In $\triangle ABC$, $B = 150°$, $c = 24$, and $a = 40$. Show that $AC = 61.95+$.



292. Right triangles whose sides are integers. From the earliest times mathematicians have been interested in methods of finding three *integers* which would represent the lengths of the sides of a right triangle. The practical importance of triangles of this sort in building construction and in rough surveying is hard to overestimate.

EXAMPLES

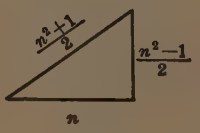
1. The rule indicated by the first triangle is that of Pythagoras, where n is any odd number.

Solution. Let $n = 5$. Then

$$\frac{n^2+1}{2} = 13 \quad \text{and} \quad \frac{n^2-1}{2} = 12.$$

Then, by § 290,

$(5)^2 + (12)^2 = (13)^2$, or $25 + 144 = 169$.



2. The second triangle indicates the rule of Plato, where m is any even number.

Solution. Let $m = 8$. Then

$$\frac{m^2}{4} - 1 = 15 \text{ and } \frac{m^2}{4} + 1 = 17.$$

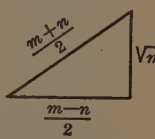
Then, by § 290,

$(8)^2 + (15)^2 = (17)^2$, or $64 + 225 = 289$.

3. The third triangle illustrates the rule of Euclid (about 330–275 B.C.), where m and n are both odd or both even.

Solution. Let $m = 9$ and $n = 25$. Then

$$\sqrt{mn} = 15, \frac{m-n}{2} = -8, \text{ and } \frac{m+n}{2} = 17.$$



Then, by § 290,

$(-8)^2 + (15)^2 = (17)^2$, or $64 + 225 = 289$.

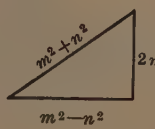
4. A rule by Masères of England (1731–1824) is illustrated last, where m and n are any two integers.

Solution. Let $m = 6$ and $n = 5$. Then

$m^2 + n^2 = 61$, $m^2 - n^2 = 11$, and $2mn = 60$.

By § 290, $(11)^2 + (60)^2 = (61)^2$,

or $121 + 3600 = 3721$.



EXERCISES

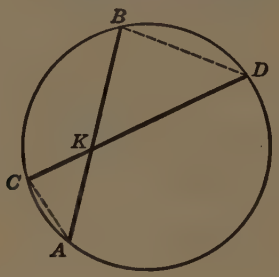
Find three integers, different from the above, which are sides of a right triangle by the following rules:

1. The rule of Pythagoras.
2. Plato's rule.
3. Euclid's rule.
4. The rule of Masères.

Plato (427–347 B.C.) was primarily a philosopher rather than a scientist or a mathematician. He emphasized in geometry the importance of definitions, clear assumptions, and logical proofs. Such axioms as "Equals subtracted from equals give equals" date from this period. His opinion of the value and importance of mathematics was very high, as is indicated by the fact that over the entrance to his school was the inscription: "Let none ignorant of geometry enter my door."

Theorem 13

293. *If two chords of a circle intersect, the product of the segments of one equals the product of the segments of the other.*



Given the chords *AB* and *CD* intersecting in point *K* within the circle.

To prove that $AK \cdot BK = CK \cdot DK$.

Method. Draw two chords joining corresponding ends of the given chords and prove that the two triangles thus formed are similar. The required proportion then follows.

Proof

1. Draw AC and BD.	1. § 51, Postulate 1.
2. $\angle A = \angle D$, and $\angle C = \angle B$.	2. § 231.
3. $\therefore \triangle AKC \sim \triangle BKD$.	3. § 276.
4. $\therefore AK : DK = CK : BK$, or $AK \cdot BK = CK \cdot DK$.	4. § 272 and § 263, 1.

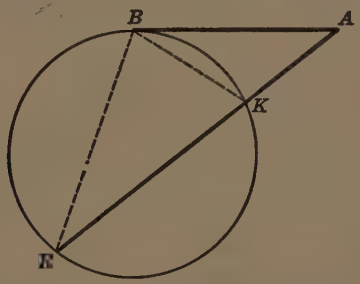
EXERCISES

In the following exercises use the figure of Theorem 13:

1. $AK = 10$, $BK = 15$, and $CK = 8$. Find DK.
2. $AK = 20$, $BK = 25$, and $CD = 60$. Find CK and DK.
3. $CK = 6$, $DK = 24$, and $AB = 26$. Find AK and BK.

Theorem 14

294. *If from a point outside a circle a secant terminating in the circle and a tangent be drawn, the square of the tangent equals the whole secant times its external segment.*



Given the secant AR, cutting the circle in K and R, and the tangent AB.

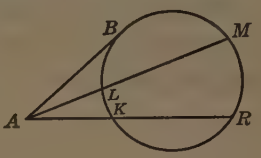
To prove that $\overline{AB}^2 = AR \cdot AK$.

Method. Draw chords joining the point of tangency to the ends of the chord of the secant. Prove that the whole triangle thus formed is similar to that of which the tangent and the external segment of the secant are sides. The required proportion then follows.

Proof

1. Draw chords BK and BR.	1. § 51, Postulate 1.
2. In $\triangle ABK$ and $\triangle ABR$ $\angle A = \angle A$.	2. Identical.
3. $\angle R = \angle ABK$.	3. §§ 229 and 235.
4. $\therefore \triangle ABK \sim \triangle ABR$.	4. § 276.
5. $\therefore \frac{AK, \text{ opposite } \angle ABK}{AB, \text{ opposite } \angle R}$ $= \frac{AB, \text{ opposite } \angle AKB}{AR, \text{ opposite } \angle ABR}$, or $\overline{AB}^2 = AR \cdot AK$.	5. § 272.

295. *Corollary.* *If from an external point two secants are drawn to a circle, one secant times its external segment equals the other secant times its external segment.*



HINT. Given ALM and AKR, the two secants. Draw the tangent AB. Then use Theorem 14.

EXERCISES

In the following exercises use the figure of Theorem 14:

1. $AK = 4$ and $KR = 5$. Show that $AB = 6$.

2. $AR = 16$ and $AB = 12$. Find AK and KR.

3. $AB = 45$ and $KR = 36$. Find AK and AR.

4. $AB = 12\sqrt{2}$ and $AK = KR$. Find AR.

5. A tangent and a secant are drawn to the circle from the same point. If the chord of the secant equals t, the length of the tangent, prove that the external segment of the secant is $\frac{t}{2}(\sqrt{5}-1)$.

6. If two circles intersect, the tangents to them from any point on their common chord produced are equal.

7. If two circles intersect, the common chord produced bisects their common tangents.

8. Points B and L divide chord FG into three equal parts. Through B and L any chords AC and KM are drawn.

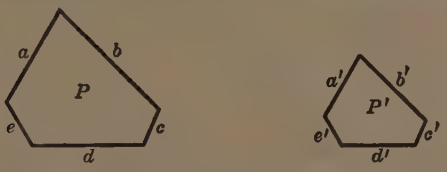
Prove $AB : KL = LM : BC$.

9. Points B and K divide chord FG into three equal parts, and $AB = AK$.

Prove $BC = KL$.

Theorem 15

296. *The perimeters of two similar polygons are in the same ratio as any two corresponding sides.*



Given the similar polygons P and P' with sides a, b, c, etc. corresponding to sides a', b', c', etc. and their perimeters p and p'.

To prove that $p : p' = a : a'$.

Method. The definition of similar polygons gives a series of equal ratios for each pair of corresponding sides. Apply the addition theorem of proportion.

Proof

1. $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{d}{d'} = \frac{e}{e'}$.	1. § 272.
2. $\therefore \frac{a+b+c+d+e}{a'+b'+c'+d'+e'} = \frac{a}{a'}$.	2. § 263, 7.
3. $\frac{p}{p'} = \frac{a}{a'}$.	3. Combining in step 2.

EXERCISES

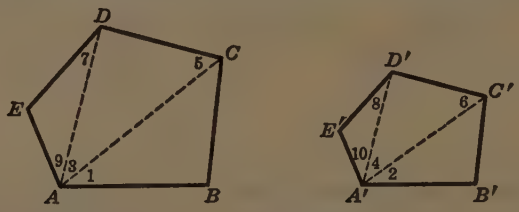
1. The sides of a polygon are 7, 12, 13, 15, and 23. The shortest side of a similar polygon is 21. Find its sides and its perimeter.

2. The perimeter of a polygon is 300. The sides of a similar polygon are 20, 30, 18, and 38. Find the sides of the first polygon.

3. The sum of the perimeters of two similar polygons is 340 ft. Two corresponding sides are 5 ft. and 12 ft. respectively. Find the perimeter of each polygon.

Construction 4

297. *Upon a given side corresponding to a side of a given polygon construct a polygon similar to the given one.*



Given the polygon *ABCDE* and the line *A'B'*.

Required to construct a polygon similar to ABCDE with AB and A'B' corresponding sides.

Method. Draw from *A* all the diagonals of the given polygon. Then construct $\angle 2 = \angle 1$, $\angle B' = \angle B$, and produce the sides of $\angle 2$ and $\angle B'$ until they meet at *C'*. Next construct $\angle 4 = \angle 3$, $\angle 6 = \angle 5$, and produce the sides of $\angle 4$ and $\angle 6$ until they meet at *D'*. Finally construct $\angle 8 = \angle 7$, $\angle 10 = \angle 9$, and produce the sides of $\angle 8$ and $\angle 10$ until they meet in *E'*. Then *A'B'C'D'E'* is the required polygon.

The proof is left for the student.

EXERCISES

1. Upon a given side corresponding to a side of a given triangle construct a triangle similar to the given one.

2. Given one side as the longer base, construct a trapezoid similar to a given one.

298. Projection. The *projection* of a line segment upon a straight line is the segment of the second line included between the perpendiculars drawn to it from the extremities of the first.

In Figs. 1, 2, and 3 are shown the projections of a line KR on a second line AB. In Fig. 1 the projection is LM,

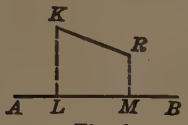


Fig. 1

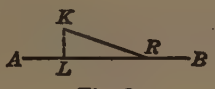


Fig. 2

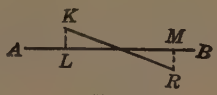


Fig. 3

in Fig. 2 it is LR, and in Fig. 3 it is LM. These correspond to the three possibilities, (a) the first line segment may not meet the second, (b) it may meet the second, (c) it may cross the second.

QUERY 1. Can the projection of a line segment be greater than the line segment itself? Can it be equal to it? Can it be less?

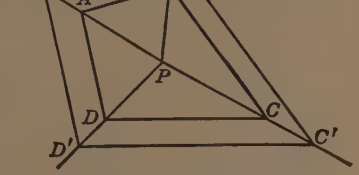
EXERCISES

1. A line 16 in. long makes an angle of 30° with another line. Show that the length of the projection of the first line on the second is $8\sqrt{3}$.

2. A line KR is 2 ft. 4 in. long and makes an angle of 45° with a line AB. Find the projection of KR on AB.

3. A line a inches long makes an angle of 60° with KR. Find the length of its projection on KR.

4. $ABCD$ is a polygon and P any point inside it. Lines PA, PB, PC, and PD are drawn through the vertices. Any point, as A', on PA extended is taken. Through A' a parallel to AB cuts PB extended in B'. Through B' a parallel to BC cuts PC extended in C'. Through C' a parallel to CD cuts PD extended in D'. Prove that polygons $ABCD$ and $A'B'C'D'$ are similar.



Theorem 16

299. *In any triangle the square of the side opposite an acute angle equals the sum of the squares of the other two sides minus twice the product of one of these sides by the projection of the other side upon it.*

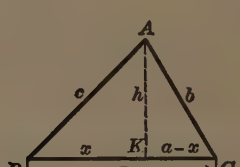


Fig. 1

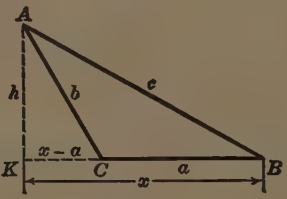


Fig. 2

Given $\triangle ABC$, in which AK is $\perp$ to BC, segment BK is the projection of side AB on side BC, and $\angle B$ is acute.

To prove that $b^2 = c^2 + a^2 - 2\,ax$.

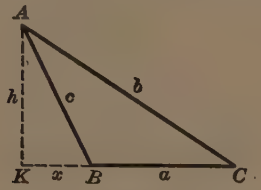
Method. Draw either altitude on either side about an acute angle and write the two equations connecting the squares of the three sides of the two right triangles thus formed. Then by algebra eliminate the altitude between these two equations, simplify, and the theorem is proved.

Proof

1. In Fig. 1, $b^2 = h^2 + (a - x)^2$.	1. § 290.
2. In Fig. 2, $b^2 = h^2 + (x - a)^2$.	2. § 290.
3. $b^2 = h^2 + x^2 - 2\,ax + a^2$.	3. Expanding in steps 1 and 2.
4. In the figures, $h^2 = c^2 - x^2$.	4. § 290.
5. $\therefore b^2 = c^2 + a^2 - 2\,ax$.	5. Substituting $c^2 - x^2$ for h^2 in step 3; § 50, Axiom 7.

Theorem 17

300. *In an obtuse-angled triangle the square of the side opposite the obtuse angle equals the sum of the squares of the other two sides plus twice the product of one of these sides by the projection of the other side upon it.*



Given $\triangle ABC$, **in which** AK **is** $\perp$ **to** BC **produced,** BK **is the projection of** BA **on** BC, **and** $\angle B$ **is obtuse.**

To prove that $b^2 = c^2 + a^2 + 2ax$.

Method. Write the two equations connecting the three sides of the two right triangles formed. Then eliminate the altitude between the two equations and simplify.

Proof

1. $b^2 = h^2 + (x + a)^2$, or $b^2 = h^2 + x^2 + 2ax + a^2$.	1. § 290.
2. $h^2 = c^2 - x^2$.	2. § 290.
3. $\therefore b^2 = c^2 + a^2 + 2ax$.	3. Substituting $c^2 - x^2$ for h^2 in step 1.

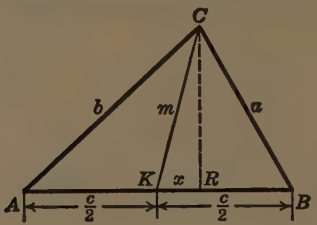
EXERCISES

1. The sides of a triangle are $a = 11$, $b = 25$, and $c = 30$. Find the projection of b on a, the altitude, and the area of the triangle.

2. Is the largest angle of the triangle whose sides are 15, 17, and 19 right, acute, or obtuse?

Theorem 18

301. *In any triangle the square of the median equals half the sum of the squares of the two adjacent sides minus the square of half the third side.*



Given $\triangle ABC$, **with** m **the median to side** AB.

To prove that $m^2 = \frac{a^2 + b^2}{2} - \frac{c^2}{4}$.

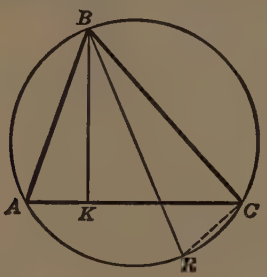
Method. Let b be greater than a and draw altitude CR. Then $\angle CKB$ is acute and $\angle CKA$ is obtuse. Write the values of a^2 and b^2 in terms of m, $\frac{c}{2}$, and the projection of m on c, add the two results, and solve for m.

Proof

1. Draw altitude CR.	1. § 76.
2. $AK = KB = \frac{c}{2}$.	2. By hypothesis.
3. In $\triangle CKB$, $a^2 = m^2 + \left(\frac{c}{2}\right)^2 - 2x\frac{c}{2}$.	3. § 299.
4. In $\triangle CKA$, $b^2 = m^2 + \left(\frac{c}{2}\right)^2 + 2x\frac{c}{2}$.	4. § 300.
5. Adding, $a^2 + b^2 = 2m^2 + \frac{c^2}{2}$.	5. § 50, Axiom 3.
6. $\therefore m^2 = \frac{a^2 + b^2}{2} - \frac{c^2}{4}$.	6. § 50, Axioms 4 and 6.

Theorem 19

302. *In any triangle the product of two sides equals the altitude on the third side multiplied by the diameter of the circumscribed circle.*



Given $\triangle ABC$, **the altitude** BK, **and** BR, **the diameter of the circumscribed circle.**

To prove that $AB \cdot BC = BK \cdot BR$.

Method. Draw the chord CR and then prove that rt. $\triangle ABK \sim$ rt. $\triangle BCR$. A proportion involving the four lines of the required equation then follows. From this proportion the required equation at once comes.

Proof

1. Draw CR.	1. § 51, Postulate 1.
2. $\angle A = \angle R$.	2. § 231.
3. $\angle BCR = 90°$.	3. § 230.
4. $\angle AKB = 90°$.	4. § 80.
5. $\therefore \triangle ABK \sim \triangle BCR$.	5. § 277.
6. $\therefore \frac{AB}{BR} = \frac{BK}{BC}$, or $AB \cdot BC = BK \cdot BR$.	6. § 272 and § 263, 1.

QUERY. If B is a right angle, how could you express this theorem so as to include the hypotenuse?

EXERCISES ON THEOREMS 17, 18, AND 19

1. In $\triangle ABC$, $a = 10$ ft., $b = 12$ ft., and $c = 16$ ft. Show that the projection of a on c is 6 ft. $7\frac{1}{2}$ in.

2. The sides of a triangle are 13, 14, and 15. Find the projection of side 13 on side 14. Then find the altitude and the area of the triangle.

3. In $\triangle ABC$, $a = 20$ ft., $b = 25$ ft., and $c = 30$ ft. Find the projection of a on b, a on c, and b on c.

4. The sides of a triangle are 11, 13, 20. Show that the median on side 20 is $\sqrt{45}$.

5. The sides of a triangle are 20, 24, and 30. Find the longest and the shortest median of the triangle.

6. Two sides and a diagonal of a parallelogram are 7, 15, and 20 respectively. Find the other diagonal.

7. The sides of a triangle are 7, 15, and 20. Show that the altitude on the side 20 is 4.2.

HINT. Draw a figure and let the altitude on side 20 be y and the segments into which it divides that side be x and $20 - x$. Then $x^2 + y^2 = 49$ and $y^2 + (20 - x)^2 = 225$. Etc.

8. Find the altitude on the side 11 of the triangle whose sides are 11, 25, and 30.

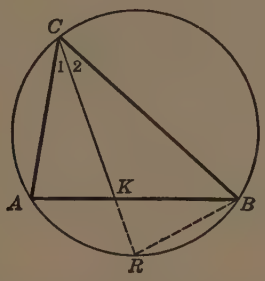
9. The sides of a triangle are 15, 28, and 41. Show that the altitude on side 28 is 9 and that the diameter of the circumscribed circle is $68\frac{1}{3}$.

10. If $\angle C$ of $\triangle ABC$ is 120°, prove that $c^2 = a^2 + b^2 + ab$.

11. The sides of a triangle in which C is an acute angle are a, b, and c respectively. Show that the projection of the side a on the side b is $\frac{a^2 + b^2 - c^2}{2b}$, and that the altitude on the side b is $\sqrt{a^2 - \left(\frac{a^2 + b^2 - c^2}{2b}\right)^2}$.

Theorem 20

303. *The product of any two sides of a triangle is equal to the square of the bisector of their included angle plus the product of the segments of the third side made by that bisector.*



Given $\triangle ABC$, **with** CK **bisecting** $\angle ACB$.

To prove that $AC \cdot CB = \overline{CK}^2 + AK \cdot KB$.

Method. Circumscribe a circle about $\triangle ABC$, extend the bisector to cut the circle at R, and draw BR. Then prove that $\triangle ACK \sim \triangle BCR$. Write the proportion involving AC, CB, CR, and CK and put it in product form. In the product form substitute $AK \cdot KB$ for $CK \cdot KR$, given by a previous theorem, and the required result follows.

Proof

1. Draw a circle about $\triangle ABC$.	1. § 174.
2. Extend CK to cut the circle in R, and draw BR.	2. § 51, Postulate 3.
3. $\angle 1 = \angle 2$.	3. By hypothesis.
4. $\angle A = \angle R$.	4. § 231.
5. Then $\triangle ACK \sim \triangle BCR$.	5. § 276.
6. Hence $CK : CB = AC : CR$.	6. § 272.

7. Then $AC \cdot CB = CK \cdot CR$ 7. § 263, 1.
$= CK(CK + KR)$
$= CK^2 + CK \cdot KR.$
8. But $CK \cdot KR = AK \cdot KB.$ 8. § 293.
9. $\therefore AC \cdot CB = \overline{CK}^2 + AK \cdot KB.$ 9. Steps 7 and 8.

EXERCISES

1. Two sides of a triangle are 10 and 12 respectively. The bisector of the angle between them divides the third side into segments 5 and 6 respectively. Show that the length of the bisector is 9.48+.

2. The sides of a triangle are 20, 21, and 29 respectively. Find the length of the bisector of the angle opposite the side 29.

3. In $\triangle ABC$, b is 24, a is 36, and c is 30. Find to one decimal the length of the bisector of $\angle B$.

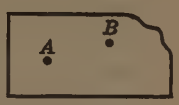
4. If a, b, and c are the sides of a triangle, show that the segments of c made by the bisector of the opposite angle are $\frac{ac}{a+b}$ and $\frac{bc}{a+b}$.

5. From Exercise 4 and Theorem 20 show that for $\triangle ABC$ the bisector of $\angle C$ is equal to $\sqrt{ab - \frac{abc^2}{(a+b)^2}}$.

304. Drawing to scale. The relations between similar polygons are widely applied in making drawings to scale. Maps, and plans for a casting, a house, or an ocean liner are all drawn to scale. This means that the drawings show figures that are similar to the actual figure they represent. For large objects the drawing is usually smaller than the object and every line in the drawing is a certain fractional part of the corresponding line in the object. On maps, one inch usually represents a considerable number of miles.

EXERCISES

1. If the scale of the adjacent map is 1 in. = 100 mi., find the distance in miles between the two cities A and B.

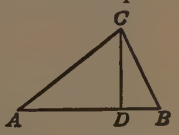


2. A building lot is 75 ft. by 144 ft. The foundation for a house is to be 36 ft. by 42 ft., the shorter dimension facing the street. The foundation must be set back 40 ft. from the front of the lot and 8 ft. from the side. Draw a plan of the lot and the foundation, making 1 in. equal 20 ft.

305. Graphical solutions of geometric problems. A very useful application of drawing to scale consists in drawing a figure to scale and then measuring some unknown line in the figure and multiplying this by the scale number to determine its real length. The given angles in the figure can be laid off by means of the protractor, and the required angles can be measured by its use.

EXAMPLE

The sides of a triangle are 24 ft., 37 ft., and 33 ft. respectively. Find the altitude on the side 37 ft.



Solution. With a scale 50 ft. = 1 in., the adjacent triangle is constructed and the altitude CD drawn. By measurement CD is found to be .4 in. long. Hence $CD = .4 \times 50 = 20$ ft. Therefore the altitude is 20 ft.

A triangle has six parts, three sides and three angles. If any three of the six parts, *of which one is a side,* is given, the other three may be found by a graphical solution. The following exercises bring out this important fact.

The scale accompanying the text is marked in inches and tenths of an inch. In using it to measure the length

of lines the decimal fractions of the tenths should be estimated as accurately as possible. This will permit reading the lengths of lines to *hundredths* of an inch. Thus, in the adjacent figure the point B is $\frac{3}{10}$ of one of the small divisions from the nearest mark on the left. Hence the length of AB is 1.73 in.

EXERCISES

Solve graphically:

1. Show that the altitude on side 18 of a triangle whose sides are 10, 16, and 18 respectively is 8.8+.

2. Find the median on the longest side of a triangle whose sides are 10, 12, and 18 respectively.

3. Find the length of the bisector of the smallest angle of the triangle whose sides are 9, 12, and 15 respectively.

4. Draw to scale a triangle whose sides are 15, 20, and 25 respectively, and find the altitude on side 15.

5. Construct to scale a right triangle in which one angle is 40° and the hypotenuse is 10. Show that the other sides are 7.6+ and 6.4+ respectively.

6. Construct a right triangle whose hypotenuse is 12 and one of whose angles is 35°. Then find the two sides.

7. Draw to scale a triangle in which one angle is 70° and the sides including it are 28 in. and 35 in. respectively. Then show that the altitude on the side 35 is 26.3+.

8. One angle of a parallelogram is 40°, and the two sides are 24 and 36 respectively. Find each diagonal.

9. One side of a triangle is 60 ft., and the two adjacent angles are 40° and 70° respectively. Find the other two sides of the triangle.

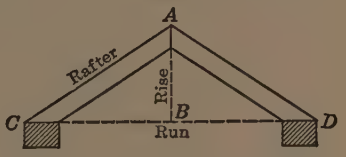
10. Two sides of a parallelogram are 35 ft. and 50 ft. respectively, and their included angle is 75°. Find the altitude of the parallelogram on the side 50 ft.

11. Construct a right triangle whose sides are 12, 16, and 20, and show that the acute angles are about 37° (36° 52′) and 53° (53° 8′) respectively.

12. The sides of a right triangle are approximately 15, 26, and 30. Find the acute angles.

13. The sides of a right triangle are 5, 12, and 13. Find the acute angles.

14. Carpenters use the word "pitch" to designate the angle the rafters of a roof make with the horizontal. It means $\frac{Rise}{Run}$, or $\frac{AB}{CD}$ of the adjacent figure. "Half pitch" means that the rise is half the run. Show that the angle corresponding to "two-thirds pitch" is 53°+ (53° 8′ very nearly).



15. Half pitch corresponds to how many degrees?

16. Draw triangles, and show that the angles corresponding to one-third pitch and one-quarter pitch are about 34° (33° 41′) and $26\frac{1}{2}$° (26° 34′) respectively.

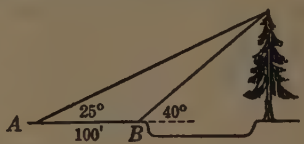
17. The sides of a triangle are 100, 125, and 150. Show that the angles are about 41°, 56°, and 83°.

18. The sides of a triangle are 19, 34, and 49. Show that the largest angle is about 133°, and find the other two angles.

19. Two sides of a triangle are 18 and 20 respectively, and their included angle is 40°. Find the third side and the other two angles.

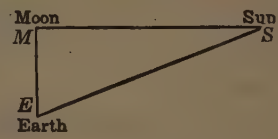
20. One side of a triangle is 36, and the adjacent angles are 40° and 65°. Find the other sides and the third angle.

21. A man standing on the bank of a river measures the angle between the horizontal and a line to the top of a tree on the opposite bank and finds it is 40°. Moving back 100 ft. on the line through his first position and the tree, he finds the angle is 25°. Show by a graphical solution that the width of the river is about 102 ft.



THE FIRST MEASURES OF THE DISTANCES OF THE SUN AND THE MOON

Aristarchus (about 250 B.C.) made the first estimate of the sun's distance from the earth by measuring $\angle E$ when the earth, moon, and sun were in such a position that $\angle EMS$ was 90°. At that time exactly half of the face of the moon turned toward the earth is illuminated by the sun. The value of $\angle E$ obtained by Aristarchus was 87°. The $\triangle MES$ could then be drawn to scale and the ratio ES to ME found by measurement. Aristarchus obtained the value $ES = 19\ ME$; that is, the sun was nineteen times as far away as the moon. Actually ES equals about 390 ME. The error in this perfectly correct method arose because of the great difficulty of measuring $\angle E$ with sufficient exactness. The correct value of $\angle E$ is 89° 50′, not 87° as found by Aristarchus.



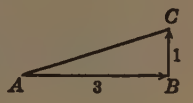
The next step forward, the finding of the moon's distance, was carried out one hundred years later by Hipparchus (about 150 B.C.). Using a better method, he determined the moon's distance as 60.5 times the earth's radius, a result very nearly correct. The value Hipparchus obtained for the distance to the sun came out twice that of Aristarchus. This was about one tenth of the real value.

306. The parallelogram law. This law, which involves a simple use of geometry, is of profound importance in physical science. Its applications today are countless. The law was discovered by Stevinus of Holland (1548–1620). Newton was the first to state it clearly. Varignon, a Frenchman, about the same time arrived independently at a definite statement of the law.

EXERCISES

1. A block of wood drifts 3 mi. per hour with the current of a river and is carried by the wind 1 mi. per hour at right angles to the current. Find the velocity of the block.

Solution. The two forces act constantly; hence the resultant motion will be along AC and the velocity will be $\sqrt{9+1} = \sqrt{10}$ mi. per hour.



2. A boat travels 10 mi. per hour. A man on its deck walks 4 mi. per hour at right angles to the direction of its motion. Find his velocity with respect to the point from which he started walking.

3. A ship is moving 8 mi. per hour. A sailor on board climbs a mast at the rate of 2 mi. per hour. Find his velocity with respect to the point at which he started climbing.

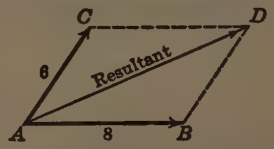
4. A river runs 3 mi. per hour. A man rows at right angles to the current 4 mi. per hour. With what velocity does he move?

5. A river runs north 3 mi. per hour. A man rows east at the same rate. What is his velocity? In what direction will he move?

307. The parallelogram of forces. The preceding exercises are simple illustrations of the parallelogram law, which applies to *forces* as well as to *velocities*.

The more general nature of the parallelogram law can be made clear as follows:

If a force of 8 lb. acts on a body A in the direction AB and at the same time a force of 6 lb. acts in the direction AC, the effect is the same as that of a single force acting along AD, the diagonal of the parallelogram $ABDC$. Moreover, if the parallelogram is drawn so that $AC:6=AB:8$, then $AC:6=AB:8=AD:x$,



where x is the force along AD. This force, x, is called the *resultant*.

In $\triangle ABD$, $\angle B$ is the supplement of the given angle, CAB. Hence the geometric problem involved here is: *Given two sides of a triangle and their included angle, find the third side of the triangle.* This problem can always be solved graphically by the methods of § 305. If the angle between the forces is 30°, 45°, or 60°, the problem can be solved numerically by the methods of § 291. If the angle between the forces is 90°, the result can be obtained by § 290.

EXERCISES

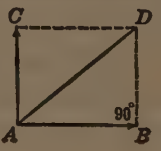
1. Two forces of 6 lb. and 8 lb., respectively, act on a body. If the angle between them is 90°, find the resultant.

2. The angle between two forces is 90°. One of the forces is 10 lb.; the other is 24 lb. Find the resultant.

3. Two forces of 42 lb. and 50 lb., respectively, act at an angle of 60°. Find the resultant.

4. The angle between two forces is 150°. The forces are 100 lb. and 80 lb. respectively. Find their resultant.

5. A ball is given a velocity of 60 ft. per second along AB and at the same instant a velocity of 50 ft. per second along AC at right angles to AB. Find the velocity along AD.

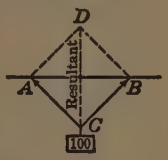


Wing-bracing in airplanes is a striking use of the parallelogram of forces

Hell Gate Bridge in New York City is made of a number of triangles arranged in the form of an arch. This gives an extremely stiff, strong bridge

6. A star has a velocity of 40 mi. per second toward the earth and a velocity of 20 mi. at right angles to the line joining earth and star. Find the resultant velocity.

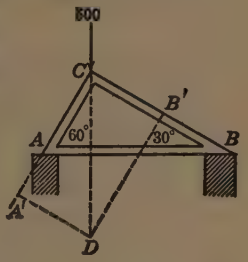
7. A weight of 100 lb. is supported by two cords, AC and BC, each of which makes an angle of 45° with the horizontal. Find the tension in AC and BC.



8. A motor car is traveling 40 mi. per hour on a road running northwest. At what rate is it moving north? west?

9. A weight of 300 lb. is supported by two equal cords, each of which makes an angle of 30° with the horizontal. Find the tension in each cord.

10. Three beams, as in the adjacent figure, support a weight of 500 lb. at C. Find the force of compression in AC and BC.



HINT. The load supported by AC and BC is proportional to CA' and CB' respectively.

11. A weight of 100 lb. is supported by two cords in such a way that the first of them makes an angle of 30° with the horizontal and the second an angle of 60°. Find the tension in each cord.

MISCELLANEOUS NUMERIC EXERCISES

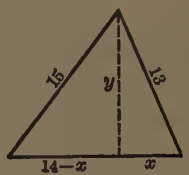
1. The sides of a right triangle are 7 and 24 respectively. Find the hypotenuse and the altitude on the hypotenuse.

2. Find the hypotenuse of a right triangle if the other two sides are $u^2 - v^2$ and $2uv$ respectively.

3. CK is the altitude on the hypotenuse AB of rt. $\triangle ABC$. If $AB = 50$ and $CK = 24$, find AK and KB.

4. AB is the hypotenuse and CK the altitude of $\triangle ABC$. If AB is 18 and AK is 8, find AC and CK.

5. In $\triangle ABC$, $a = 13$, $b = 14$, and $c = 15$. Show that the altitude on side 14 is 12.



HINT. $y^2 = (13)^2 - x^2$ and $y^2 = (15)^2 - (14-x)^2$.

6. Show that the altitude on the side 9 of the triangle whose sides are 9, 10, and 17 is 8.

7. A chord 24 in. long is 5 in. distant from the center. Find the radius of the circle.

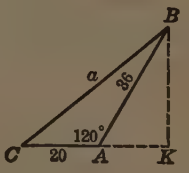
8. The sides of a triangle are 18, 20, and 22. Find the segments of side 20 made by the bisector of the opposite angle.

9. In $\triangle ABC$, $\angle A = 60°$, $c = 16$, and $b = 6$. Find a.

10. The shadow of a church steeple upon level ground is 120 ft. At the same time a pole 10 ft. high casts a shadow 8 ft. 4 in. long. Find the height of the steeple.

11. The radius of a circle is 12 in. Through a point 4 in. from the center a chord is drawn. Find the product of the segments of the chord.

12. In $\triangle ABC$, $\angle A = 120°$, $b = 20$, and $c = 36$. Find a.



13. In $\triangle ABC$ $a = 12$, $b = 17$, and $c = 25$. Find the altitude on a.

14. The nonparallel sides, AD and BC, of a trapezoid are produced to meet in K. If $AB = 16$, $CD = 12$, and $AD = 8$, find AK and DK.

15. The bases of a trapezoid are a and b and the altitude is h. The nonparallel sides are produced until they meet. Find the altitude of each triangle thus formed.

16. The altitude of an equilateral triangle is 12. Find to three decimals the length of one side.

17. ABC and AKR are secants to the same circle. AB is 8, chord BC is 10, and chord KR is 24. Find AK and the tangent to the circle from A.

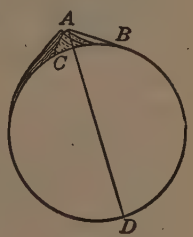
18. The radii of two circles are 12 and 20 respectively. The distance between their centers is 40. How far from the center of the smaller circle does (*a*) the external common tangent cut the line of centers extended? (*b*) the internal tangent cut the line of centers?

19. AB and KR are two chords of a circle intersecting at O. AB is 30, OR is 24, and OK is 9. Find AO and OB.

20. A line AB, 12 in. long, is divided externally in the ratio of 2 : 5. Find the segments of AB.

21. The hypotenuse of a right triangle is 29 and another side is 21. The perimeter of a similar triangle is 280. Find the shortest side of the second triangle.

22. On a clear day a certain cliff can be seen from a point at sea 48 mi. away. How high is the cliff?



Solution. Often, as here, one or more terms in the solution of a problem may be neglected without seriously affecting the result. This usually shortens the work and frequently gives results close enough for the purpose required.

Let $AC = x$ and $CD = 2r$ ($r = 3960$ mi.).

Then $$\overline{AB}^2 = x(x + 2r) = 2rx + x^2.$$

Since even Mount Everest is less than 6 mi. high x^2 here is very small compared to $2rx$. Neglecting the term x^2 avoids the solution of a quadratic equation and gives

$$x = \frac{\overline{AB}^2}{2r} = \frac{(48)^2}{7920} = \frac{16}{55} = .2909 \text{ mi.}$$

23. How high is a mountain if a boatman 30 mi. away can just see its top from the surface of the ocean?

24. How far is a swimmer from the top of a cliff 400 ft. high if when his eye is level with the surface of the water the top of the cliff is just visible? (Use 3960 mi. for the earth's radius.)

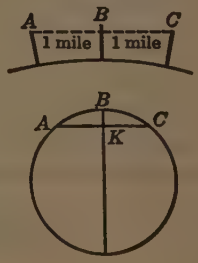
25. Astronomers can measure the lengths of the shadows cast by the mountains on the moon. This was first done by Galileo some time after 1610, when he began using the newly invented telescope for astronomical observations. The longest shadow cast by one of the highest mountains on the moon has been found to be 90 mi. long. If the moon's diameter is 2160 mi., find the height of the mountain.

26. A lookout 100 ft. above the surface of the ocean observes an object close to the surface as it comes into view on the edge of the horizon. How far away is the object?

27. One of two forces which act at right angles to each other is 24 lb. and the resultant of the two is 30 lb. Show that the other force is 18 lb.

28. How high must an aviator rise to be able to see 40 mi. in every direction over a smooth sea?

29. On the smooth ice of a lake three vertical standards, each 5 ft. long, were set up in a straight line at intervals of 1 mi. On looking through a telescope across the top of the first standard to the top of the third it was found that the line of sight fell 8 in. below the top of the second standard. Why? Show from the preceding facts that the diameter of the earth is 7920 mi.



CONSTRUCTIONS INVOLVING PROPORTION

1. Given the perimeter, construct a triangle similar to a given triangle.

HINT. Use Construction 2, § 286.

2. Given the perimeter, construct a rectangle similar to a given rectangle.

3. Divide one side of a triangle into two parts proportional to the other two sides.

4. Construct a triangle, given two sides, a and b, and the ratio of a to the third side c, expressed by two other given lines, h and k.

HINT. Find a fourth proportional to h, k, and a.

5. Construct a triangle, given one side a, $h:k$ the ratio of a to b, and $m:n$ the ratio of a to c.

6. Draw a line parallel to one side of a rectangle, cutting off a rectangle similar to the given one.

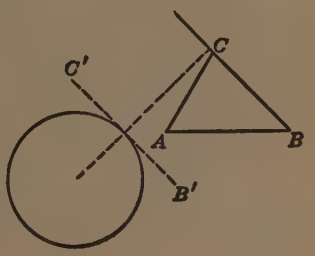
HINT. In the figure $a:b=b:x$.

7. Inscribe in a given circle a triangle similar to a given one.

HINT. Circumscribe a circle about the given triangle. Join its center to the vertices and study the central angles so formed.

8. Circumscribe about a given circle a triangle similar to a given one.

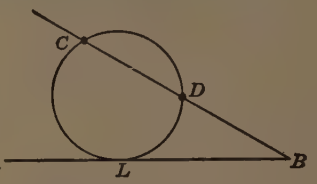
HINT. How can $B'C'$ be drawn tangent to the circle and parallel to BC?



9. Construct a trapezoid similar to but not equal to a given one.

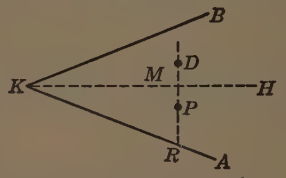
10. Given secant AC from A cutting a circle in B and C. Construct a line equal in length to the tangent from A.

11. Construct a circle touching a given line and passing through two fixed points.



HINT. Let AB be the given line and C and D the given points. How can the length of BL and the position of L be determined?

12. Construct a circle touching two fixed lines and passing through a given point.

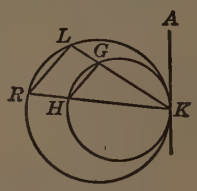


HINT. Let KA and KB be the given lines and P the given point. Draw KH bisecting $\angle AKB$. Draw $PM \perp$ to KH, meeting it at M. Extend PM to D, making $MD = PM$. Let MP cut AK in R. Then, if T is the required point of tangency for KA, $\overline{RT}^2 = RP \cdot RD$.

MISCELLANEOUS THEOREMS

1. If two circles are tangent, the chords formed by a line through the point of contact have the ratio of the diameters.

2. Two circles touch internally at K. Two chords, KL and KR, of the greater circle cut the smaller circle in G and H respectively. Draw chords LR and GH.



Prove $\triangle KGH \sim \triangle KLR$.

3. AB is a diameter of a circle. Through A any straight line is drawn cutting the circle in C and the tangent at B in K.

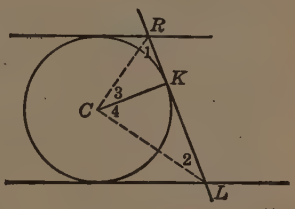
Prove $AC : AB = AB : AK$.

4. A common internal tangent to two unequal circles divides their line of centers into two segments which are proportional to the diameters of the circles.

5. CK is the bisector of $\angle C$ of $\triangle ABC$. $\angle A = \angle ACK$.

Prove $AC : CK = AB : CB$.

6. A tangent to a circle at K cuts two parallel tangents in R and L.



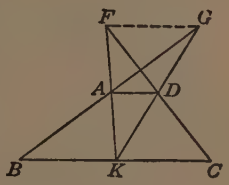
Prove that the radius of the circle is a mean proportional between KR and KL.

HINT. How many degrees in $\angle 1 + \angle 2$? in $\angle 3 + \angle 4$?

7. K is the mid-point of arc AB. The chords KR and KL cut the chord AB in H and G respectively. Draw chord RL.

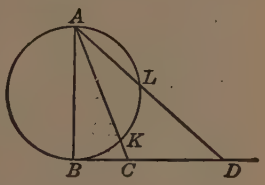
Prove $\triangle KHG \sim \triangle KRL$.

8. K is the mid-point of BC, one of the parallel sides of trapezoid $ABCD$. The lines KA and KD produced meet CD and BA produced in F and G respectively.



Prove FG is $\parallel$ to AD.

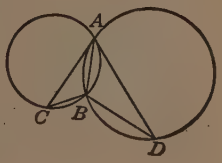
9. AB is a diameter. Chords AL and AK meet the tangent to the circle at B in D and C respectively.



Prove $AK \cdot AC = AL \cdot AD$.

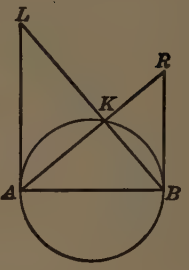
10. If an isosceles triangle is inscribed in a circle, tangents at its vertices form another isosceles triangle.

11. AB is the common chord of two circles, and chords AC and AD are tangents to the circles.



Prove AB is a mean proportional between BC and BD.

12. K is any point on the semicircle AB. AK cuts the tangent at B in R, and BK cuts the tangent at A in L.

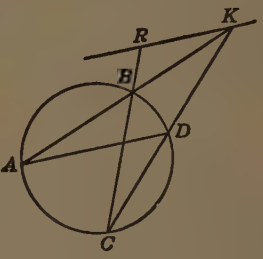


Prove that AB is a mean proportional between AL and BR.

13. From the vertex D of the $\square ABCD$ a straight line is drawn cutting AB at R and CB produced at K.

Prove that CK is a fourth proportional to AR, AD, and AB.

14. Two chords, AB and CD, of a circle are produced to meet in K. Then KR is drawn parallel to AD, meeting CB produced at R.



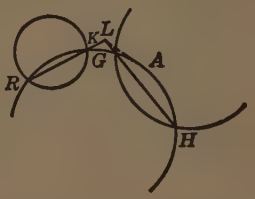
Prove $\overline{KR}^2 = BR \cdot CR$.

15. If two lines parallel respectively to two sides of a triangle are drawn through the point of intersection of the medians, they trisect the third side. Prove.

16. Through K, any point in the common base of the $\triangle ABC$ and ABR, lines are drawn parallel to AC and AR, meeting BC and BR in the points F and G respectively.

Prove $\triangle ACR \sim \triangle KFG$.

17. Two nonintersecting circles are cut by a third circle. The two common chords RK and HG intersect at L. Secant LKR cuts the first circle in K and R, and secant LGH cuts the second in G and H.

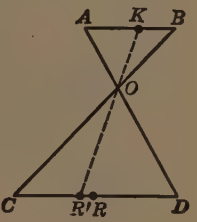


Prove that $LK \cdot LR = LG \cdot LH$ and that the tangents from L to the three circles are equal.

HINT. Draw a tangent LA to the third circle.

18. Two parallel lines, AB and CD, are joined by AD and BC, which intersect at O. On AB and CD points K and R are taken so that $AK : KB = DR : RC$.

Prove that K, O, and R are in a straight line.



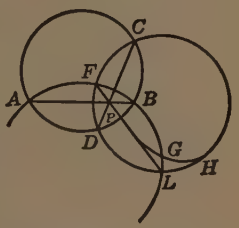
HINT. Draw KO and suppose it cuts CD not in R but in R'. R' can be proved to coincide with R as follows:

$$AK : KB = DR' : CR'.$$
$$\therefore AK : AB = DR' : CD.$$

By hypothesis, $AK : KB = DR : CR$.

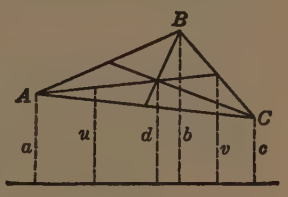
$$\therefore AK : AB = DR : CD.$$
$$\therefore DR' = DR, \text{ etc.}$$

19. If each of three circles intersects the other two, the three common chords meet at a point.



HINT. Let AB and CD intersect at P. Let FP cut the circles at G and H. Then use § 293 and prove that G and H must coincide with L.

20. The distance of the point of intersection of the medians of a triangle from any line is one third the sum of the perpendiculars from the vertices of the triangle to the line.



HINT. $v = \dfrac{b+c}{2}$. (1)

$d = \dfrac{u+v}{2}$. (2)

$u = \dfrac{a+d}{2}$. (3)

Eliminate u and v from (1), (2), and (3) and then solve for d.

21. A common external tangent to two unequal circles divides the line of centers, produced, into segments proportional to the radii.

Trigonometric Ratios

308. Introduction to trigonometry. Every triangle has six parts, three sides and three angles. It was proved in Book I that two triangles are congruent if both have the following equal parts:

(a) two sides and their included angle;
(b) a side and two adjacent angles;
(c) three sides.

All this may be summarized as follows: Given three parts of a triangle, as specified under (a), (b), or (c), then the values of the other three parts are determined by the three given ones.

The method of computing the other parts of a triangle when the three parts just described are known is part of the work of that branch of mathematics called *trigonometry*. As will be seen, it involves an important application of similar triangles. Usually this computation of the other parts by means of the given or measured parts is called *solving* the triangle. Trigonometry was developed in ancient times by the astronomers who wished to solve certain astronomical problems. One of these is described on page 286. It will be found that the method of solution is numeric and totally different from the graphical method of § 305. Incidentally, the trigonometric method can be more accurate than the graphical method.

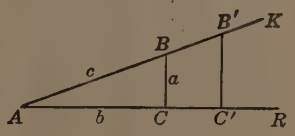
309. The trigonometric functions. In § 291 is found the numeric solution of a right triangle in which one side is given and one acute angle is 30°, 45°, or 60°. In order to solve right triangles in which the angles may have any possible values, it is necessary to have for each angle a certain ratio. These ratios are called the trigonometric *ratios* or *functions* and are defined as follows:

Let KAR be any angle. From any point on either side, as AK, draw BC a perpendicular to the other side. Let $BC = a$, $AC = b$, and $AB = c$. Then

$$\textit{sine } \angle A = \frac{a}{c} = \frac{\text{side opposite}}{\text{hypotenuse}},$$

$$\textit{cosine } \angle A = \frac{b}{c} = \frac{\text{side adjacent}}{\text{hypotenuse}},$$

$$\textit{tangent } \angle A = \frac{a}{b} = \frac{\text{side opposite}}{\text{side adjacent}},$$



and $\textit{cotangent } \angle A = \frac{b}{a} = \frac{\text{side adjacent}}{\text{side opposite}}$.

The foregoing ratios are commonly written as $\sin A = \frac{a}{c}$, $\cos A = \frac{b}{c}$, $\tan A = \frac{a}{b}$, and $\cot A = \frac{b}{a}$ respectively.

It is important to observe that the value of each ratio depends wholly on the size of $\angle A$, not on the size of $\triangle ABC$. Thus, in the figure above,

$$\sin A = \frac{CB}{AB} = \frac{C'B'}{AB'}.$$

It is well for the student to remember that trigonometric ratios are not measures of distance, and consequently cannot be expressed in feet, inches, or any other similar unit. The numbers given in a trigonometric table are ratios, pure and simple, and can be used in connection with any system of length measurement. The values of trigonometric ratios to four decimal places for every angle from 1° to 90° are given in the table on page 300.

310. The trigonometric functions of 60°. The meaning of the table is best understood by computing the ratios for 30°, 45°, or 60°. The computation for an angle of 60° is given on page 301.

Four-place Trigonometric Table

Angle	sin opp./hyp.	cos adj./hyp.	tan opp./adj.	cot adj./opp.	Angle	sin opp./hyp.	cos adj./hyp.	tan opp./adj.	cot adj./opp.
0°	.0000	1.0000	.0000	∞	45°	.7071	.7071	1.0000	1.0000
1°	.0175	.9998	.0175	57.290	46°	.7193	.6947	1.0355	.9657
2°	.0349	.9994	.0349	28.636	47°	.7314	.6820	1.0724	.9325
3°	.0523	.9986	.0524	19.081	48°	.7431	.6691	1.1106	.9004
4°	.0698	.9976	.0699	14.300	49°	.7547	.6561	1.1504	.8693
5°	.0872	.9962	.0875	11.430	50°	.7660	.6428	1.1918	.8391
6°	.1045	.9945	.1051	9.5144	51°	.7771	.6293	1.2349	.8098
7°	.1219	.9925	.1228	8.1443	52°	.7880	.6157	1.2799	.7813
8°	.1392	.9903	.1405	7.1154	53°	.7986	.6018	1.3270	.7536
9°	.1564	.9877	.1584	6.3138	54°	.8090	.5878	1.3764	.7265
10°	.1736	.9848	.1763	5.6713	55°	.8192	.5736	1.4281	.7002
11°	.1908	.9816	.1944	5.1446	56°	.8290	.5592	1.4826	.6745
12°	.2079	.9781	.2126	4.7046	57°	.8387	.5446	1.5399	.6494
13°	.2250	.9744	.2309	4.3315	58°	.8480	.5299	1.6003	.6249
14°	.2419	.9703	.2493	4.0108	59°	.8572	.5150	1.6643	.6009
15°	.2588	.9659	.2679	3.7321	60°	.8660	.5000	1.7321	.5774
16°	.2756	.9613	.2867	3.4874	61°	.8746	.4848	1.8040	.5543
17°	.2924	.9563	.3057	3.2709	62°	.8829	.4695	1.8807	.5317
18°	.3090	.9511	.3249	3.0777	63°	.8910	.4540	1.9626	.5095
19°	.3256	.9455	.3443	2.9042	64°	.8988	.4384	2.0503	.4877
20°	.3420	.9397	.3640	2.7475	65°	.9063	.4226	2.1445	.4663
21°	.3584	.9336	.3839	2.6051	66°	.9135	.4067	2.2460	.4452
22°	.3746	.9272	.4040	2.4751	67°	.9205	.3907	2.3559	.4245
23°	.3907	.9205	.4245	2.3559	68°	.9272	.3746	2.4751	.4040
24°	.4067	.9135	.4452	2.2460	69°	.9336	.3584	2.6051	.3839
25°	.4226	.9063	.4663	2.1445	70°	.9397	.3420	2.7475	.3640
26°	.4384	.8988	.4877	2.0503	71°	.9455	.3256	2.9042	.3443
27°	.4540	.8910	.5095	1.9626	72°	.9511	.3090	3.0777	.3249
28°	.4695	.8829	.5317	1.8807	73°	.9563	.2924	3.2709	.3057
29°	.4848	.8746	.5543	1.8040	74°	.9613	.2756	3.4874	.2867
30°	.5000	.8660	.5774	1.7321	75°	.9659	.2588	3.7321	.2679
31°	.5150	.8572	.6009	1.6643	76°	.9703	.2419	4.0108	.2493
32°	.5299	.8480	.6249	1.6003	77°	.9744	.2250	4.3315	.2309
33°	.5446	.8387	.6494	1.5399	78°	.9781	.2079	4.7046	.2126
34°	.5592	.8290	.6745	1.4826	79°	.9816	.1908	5.1446	.1944
35°	.5736	.8192	.7002	1.4281	80°	.9848	.1736	5.6713	.1763
36°	.5878	.8090	.7265	1.3764	81°	.9877	.1564	6.3138	.1584
37°	.6018	.7986	.7536	1.3270	82°	.9903	.1392	7.1154	.1405
38°	.6157	.7880	.7813	1.2799	83°	.9925	.1219	8.1443	.1228
39°	.6293	.7771	.8098	1.2349	84°	.9945	.1045	9.5144	.1051
40°	.6428	.7660	.8391	1.1918	85°	.9962	.0872	11.430	.0875
41°	.6561	.7547	.8693	1.1504	86°	.9976	.0698	14.300	.0699
42°	.6691	.7431	.9004	1.1106	87°	.9986	.0523	19.081	.0524
43°	.6820	.7314	.9325	1.0724	88°	.9994	.0349	28.636	.0349
44°	.6947	.7193	.9657	1.0355	89°	.9998	.0175	57.290	.0175
45°	.7071	.7071	1.0000	1.0000	90°	1.0000	.0000	∞	.0000

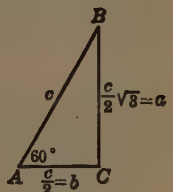
From § 291, $b = \frac{c}{2}$ and $a = \frac{c}{2}\sqrt{3}$. Therefore

$$\sin 60° = \frac{\frac{c}{2}\sqrt{3}}{c} = \tfrac{1}{2}\sqrt{3} = .866+,$$

$$\cos 60° = \frac{\frac{c}{2}}{c} = \frac{1}{2} = .500,$$

$$\tan 60° = \frac{\frac{c}{2}\sqrt{3}}{\frac{c}{2}} = \sqrt{3} = 1.732+,$$

$$\cot 60° = \frac{\frac{c}{2}}{\frac{c}{2}\sqrt{3}} = \frac{1}{\sqrt{3}} = .577.$$



The ratios for 30° and 45° can be worked out with equal ease. It is not possible, by simple means, to compute any other ratios except those for 0°, 30°, 45°, 60°, and 90°.

311. Graphical method of computing the functions of any angle. If graphical methods are used, however, the functions of any acute angle are readily obtained, as follows:

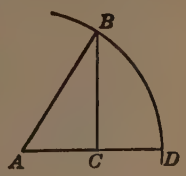
Using a circle of two-inch radius, lay off with the protractor an arc $BD = 58°$, for example. Draw AB and then draw BC perpendicular to AD. Then by measurement find $AC = 1.06$ in. and $BC = 1.70$ in.

$$\therefore \sin 58° = \frac{1.70}{2} = .850, \quad (.8480)$$

$$\cos 58° = \frac{1.06}{2} = .530, \quad (.5299)$$

$$\tan 58° = \frac{1.70}{1.06} = 1.603, \quad (1.6003)$$

$$\cot 58° = \frac{1.06}{1.70} = .623. \quad (.6249)$$



The accuracy of graphical methods here is limited even if large-scale figures are drawn. Hence in practice it is found best to use the tables of the ratios which have been computed by other means, in which values to four or more decimals are given. The values of the functions of 58°, correct to four places, are given in the parentheses.

EXERCISES

1. Read from the table the following:

(*a*) sin 32°.	(*d*) cot 24°.	(*g*) tan 81°.
(*b*) tan 44°.	(*e*) sin 54°.	(*h*) cot 75°.
(*c*) cos 62°.	(*f*) cos 78°	(*i*) cos 28°.

2. From the table determine the number of degrees in x if

(*a*) $\sin x = .1736$.	(*e*) $\tan x = .3839$.	(*i*) $\sin x = .9877$.
(*b*) $\sin x = .9744$.	(*f*) $\tan x = 1.4281$.	(*k*) $\cos x = .1564$.
(*c*) $\cos x = .2756$.	(*g*) $\cot x = .6249$.	(*l*) $\tan x = .1228$.
(*d*) $\cos x = .9135$.	(*h*) $\cot x = 3.0777$.	(*m*) $\cot x = .4040$.

3. Construct a right triangle having one angle 70°. Measure a, b, and c and compute the four functions of 70°. Compare results with those in the table.

4. Compute as in § 310 the four functions of 30°, giving inexact results in radical form.

5. Compute as in § 310 the four functions of 45°.

6. Tabulate the results of § 310 and Exercises 4 and 5 in tabular form, as follows:

	30°	45°	60°
Sine	?	?	?
Cosine	?	?	?
Tangent	?	?	?
Cotangent	?	?	?

7. The sides of a right triangle are 3, 4, and 5. Find the four functions of its smaller acute angle.

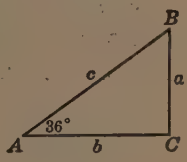
8. As an angle increases does its sine increase? its tangent? Examine the sines and tangents in the table.

9. Examine the cosines and cotangents in the table and determine if they increase or decrease as the angle increases.

312. Trigonometric solution of triangles. The trigonometric functions can be used to solve any triangle. It is easier, however, to begin with right triangles, and the simplest cases even of them.

EXAMPLE

In $\triangle ABC$, $\angle A$ is 36° and $AB = 52$. Find AC and BC.



Solution. $\sin A = \frac{a}{c}$,

or $a = c \sin A = c \sin 36°$

$= 52(.5878) = 30.56.$

Also, $\cos A = \frac{b}{c}$,

or $b = c \cos A = c \cos 36°$

$= 52(.8090) = 42.068.$

EXERCISES

In the following the triangle is lettered as in § 312.

1. $A = 42°$, $c = 30$. Find a, b, and B.

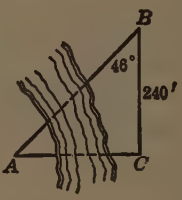
2. $B = 76°$, $c = 120$. Find the other parts.

3. $A = 40°$, $b = 60$. Find the other parts.

HINT. $\tan A = \frac{a}{b}$, or $a = b \tan 40°$.

4. $B = 65°$, $a = 80$. Find the other parts.

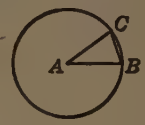
5. The shadow of a certain tree is 120 ft. long. At the same time the sun's rays make an angle of 40° with the ground. Find the height of the tree.



6. In the figure, $C = 90°$, $B = 46°$, and $BC = 240$ ft. Find AC.

7. The base of an isosceles triangle is 20 and the vertex angle is 56°. Find the equal sides of the triangle.

8. One of the equal sides of an isosceles triangle is 84 ft. and the vertex angle is 62°. Find the base of the triangle.



9. In the adjacent figure the radius $AB = 100$ and BC is the side of a regular decagon. Find BC.

313. Extension of the tables; interpolation. In order to solve more difficult problems with right triangles, it is necessary to learn two additional uses of the table.

(*a*) Given an *angle* not in the table, to find the corresponding sine, cosine, tangent, or cotangent.

(*b*) Given the *value* of a sine, cosine, tangent, or cotangent not in the table, to find the corresponding angle.

The carrying out of either of these two steps is called *interpolation*. It means finding from numbers given in the table the value of an angle or a function not given there. A fundamental principle underlying the two problems (*a*) and (*b*) is that

(*c*) as the angle increases the *sine* and the *tangent increase*;

(*d*) as the angle increases the *cosine* and the *cotangent decrease*.

EXAMPLES

1. Find the sine of 25° 20′.

Solution. $\left.\begin{array}{l}\sin 25° = .4226.\\ \sin 26° = .4384.\end{array}\right\}$ Difference for one degree is .0158.

$20' = \frac{20}{60} = \frac{1}{3}$ of one degree. $\quad \frac{1}{3}(.0158) = .0053.$

Since the sine increases as the angle increases, this must be added to sin 25°. Therefore

$$\sin 25° 20' = .4226 + .0053 = .4279.$$

2. Find the cosine of 62° 24′.

Solution. $\left.\begin{array}{l}\cos 62° = .4695.\\ \cos 63° = .4540.\end{array}\right\}$ Difference for one degree is .0155.

$24' = \frac{24}{60} = .4$ of a degree. $\quad .4(.0155) = .00620.$

Since the cosine decreases as the angle increases, .0062 must be subtracted from cos 62°; that is,

$$\cos 62° 24' = .4695 - .0062 = .4633.$$

3. Find x if $\sin x = .3638$.

Solution. The sines of 21° and 22° are respectively less and greater than the given sine. Therefore the value of x lies between 21° and 22°.

$\left.\begin{array}{l}\sin 21° = .3584.\\ \sin x \;\; = .3638.\\ \sin 22° = .3746.\end{array}\right\}$ Difference is .0054. $\quad$ Difference for one degree is .0162.

$$\frac{.0054}{.0162} \times 60' = \frac{1}{3} \times 60' = 20'. \qquad \therefore x = 21° 20'.$$

4. Find x if $\cos x = .3469$.

Solution. $\left.\begin{array}{l}\cos 69° = .3584.\\ \cos x \;\; = .3469.\\ \cos 70° = .3420.\end{array}\right\}$ Difference is .0115. $\quad$ Difference for one degree is .0164.

$$\frac{.0115}{.0164} \times 60' = 42'. \qquad \therefore \sin x = 69° 42'.$$

The process of interpolation is illustrated in these examples. The work for the *tangent* is carried out precisely as for the *sine* and the work for the *cotangent* precisely as for the *cosine*.

EXERCISES

Find x by interpolation if

1. $\sin x = .1435$.
2. $\tan x = .2035$.
3. $\sin x = .8962$.
4. $\tan x = 1.3599$.
5. $\cos x = .9304$.
6. $\cos x = .8125$.
7. $\cot x = 1.1780$.
8. $\cot x = .4383$.
9. $\sin x = .1846$.
10. $\tan x = 1.4974$.

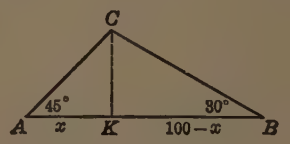
Find the angles of the right triangles of which one side and the hypotenuse are as follows:

11. 5, 4.
12. 13, 12.
13. 17, 8.

HINT. $\sin A = \frac{4}{5} = .8$. Etc.

14. In the $\triangle ABC$ side $AB = 100$, $\angle A = 45°$, and $\angle B = 30°$. Find AC and BC.

HINT. Let $AK = x$.
Then $BK = 100 - x$ and $CK = x$.
Then $\dfrac{x}{100 - x} = \tan 30°$. Etc.



15. The sides of an equilateral triangle are each 36 in. Find the radius of its inscribed circle.

16. The sides of an equilateral triangle are each 60. Find the radius of its circumscribed circle.

17. What angle do the sun's rays make with the horizontal at a time when a flagpole 40 ft. high casts a shadow 60 ft. long?

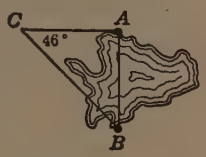
18. In $\triangle ABC$ side $AB = 32$, $AC = 20$, and $\angle A = 46°$. Find the altitude.

19. Find the altitude on AB of a $\square ABCD$ if $AB = 60$, $AD = 40$, and $\angle A = 65°$.

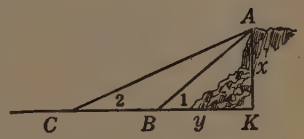
20. The radius of a circle is 10 ft. A chord of the circle has an arc of 40°. Find the length of the chord.

21. $ABCD$ is a trapezoid in which the parallel sides AB and CD are 100 and 80 respectively, $AD = 60$ and $\angle A = 70°$. Find the altitude of the trapezoid and BC.

22. In order to find the distance AB across a lake, AC is run 250 yd. perpendicular to AB and $\angle ACB$ is found to be 46°. Find AB.



23. A cliff AK stands in a level valley. In order to determine its height two points, C and B, are taken in a straight line with K, the base of the cliff. CB is measured and found to be 800 yd. and $\angle 1 = 40°$ and $\angle 2 = 24°$. Find AK.

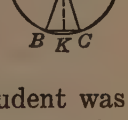


HINT. Let $AK = x$ and $BK = y$.

Then $\frac{x}{y} = \tan \angle 1$ and $\frac{x}{800 + y} = \tan \angle 2$.

Substitute the values of $\tan \angle 1$ and $\tan \angle 2$ in these equations and solve for x, etc.

24. Find the side of a regular decagon inscribed in a circle whose radius is 10 ft.

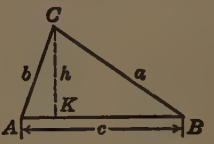


HINT. How many degrees in $\angle ACK$? How can KC be found? BC?

314. Area of a triangle. In arithmetic the student was told that the area of a triangle was equal to one half the product of the base and the altitude of the triangle. This relationship is usually expressed by the formula $A = \frac{ab}{2}$, where A is the area of the triangle, a is the altitude upon side b, and b is the base. Although geometric proof of this statement is not given until later, we may make use of the fact in the exercises on the following page.

EXERCISES

1. Find the area of a triangle given two sides and their included angle.



Solution. The area of $\triangle ABC = \frac{hc}{2}$.

From $\triangle ACK$, $\frac{h}{b} = \sin A$ or $h = b \sin A$.

$\therefore$ area of $\triangle ABC = \frac{(b \sin A)c}{2} = \frac{bc \sin A}{2}$.

2. Two sides of a triangle are 10 ft. and 12 ft. respectively and their included angle is 42°. Find the area.

3. The equal sides of an isosceles triangle are each 24 ft. and the vertex angle is 74°. Find the area of the triangle.

4. Find the area of a regular pentagon inscribed in a circle whose radius is 12 ft.

HINT. How can the area of $\triangle ABC$ be found?

The importance of Theorems 16 and 17 in trigonometry and physics is indicated by the following exercises:

5. In Theorem 16 show that $x = c \cos B$. Then prove that the result of that theorem, $b^2 = a^2 + c^2 - 2ax$, becomes $b^2 = a^2 + c^2 - 2ac \cos B$. Solve this result for $\cos B$.

6. In the same way show that $c^2 = a^2 + b^2 - 2ab \cos C$. Solve this result for $\cos C$.

7. Show how the three angles of a triangle can be found by the two preceding results even if one angle is obtuse.

8. The sides of a triangle are $a = 10$, $b = 12$, $c = 14$. Show that $\angle A = 44° 25'$, $\angle B = 57°$, and $\angle C = 78° 28'$ approximately.

9. By the use of Theorems 16 and 17 it can be shown that the resultant of forces a and b acting at an angle C is $R = \sqrt{a^2 + b^2 + 2ab \cos C}$. Two forces of 10 lb. and 12 lb. act at 59°. Show that their resultant is about 19.1.

REVIEW QUESTIONS

1. What is a ratio? a proportion?

2. How can the truth of a given proportion be tested?

Form a proportion from the following:

3. $3xy = am$. 4. $c = \frac{ax}{m}$. 5. $m = ac$. 6. $2 = ab$.

7. What are similar triangles? similar polygons?

8. What is the ratio of the side of a square to the diagonal?

9. What are commensurable lines? Give an example.

10. What are incommensurable lines? Give an example.

11. How are corresponding sides of similar triangles determined? corresponding sides of similar polygons?

12. Can two dissimilar polygons have corresponding sides? Can their sides be proportional?

13. Are two regular octagons similar?

14. Are two regular polygons similar? Why?

15. Are congruent polygons similar?

16. If the sides of two polygons taken in order are proportional, are they similar? What is the answer if the polygons are triangles?

17. If two polygons are mutually equiangular, are they similar? What is the answer if the polygons are triangles?

18. What is true of corresponding lines of similar figures?

19. State five conditions for the similarity of two triangles.

20. What is the Pythagorean Theorem?

21. What relation exists between the segments of two intersecting chords of a circle?

22. State the theorem concerning a tangent and a secant from a point outside a circle.

23. State the theorem concerning two secants from a point outside a circle.

24. AB is a line and P is a point on the line and K a point on AB produced. Name the segments for internal division of the line; for external division.

25. State the theorem concerning the segments of one side of a triangle made by the bisector of the angle opposite.

26. State the corresponding theorem for the bisector of the exterior angle opposite.

27. On what property of similar figures does map-making depend? scale-drawing?

28. Draw two similar quadrilaterals (general ones) with their corresponding sides parallel. Select a point within one and join it to the vertices. Can a point be found in the other quadrilateral such that lines to its vertices will give four triangles similar respectively to the first four?

29. Two polygons are similar. Can they be divided into similar triangles by any other lines than their diagonals?

30. Given one side and two adjacent angles of a triangle, explain how the other parts can be found by drawing to scale.

31. Answer question 30 if given (*a*) the three sides of a triangle; (*b*) two sides and the included angle.

32. What *geometric* problem is involved in any application of the parallelogram law of forces or velocities?

Summary

(Black figures refer to sections)

315. Résumé of Book III. 1. *Definitions.* Ratio, 255; measurement, 256; commensurable magnitudes, 257; incommensurable magnitudes, 258; proportion, 259; fourth proportional, 261; mean proportional, 262; segments of a line made by a point, 270; similar polygons, 272; corresponding angles and sides, 273.

2. *Algebraic theorems of proportion.* Alternation, inversion, addition, subtraction, general addition theorem, 263.

3. *Theorems on lines dividing one or more sides of a triangle proportionally.* Lines parallel to one side of a triangle, 264, 265; segments of two lines made by three or more parallels, 266; line dividing two sides of a triangle proportionally, 268; segments of one side of a triangle made by bisector of the angle opposite, 269; segments of side of a triangle made by bisector of exterior angle, 271.

4. *Theorems on similar triangles.* Mutually equiangular triangles are similar, 275; two triangles are similar if two angles of one equal two angles of the other, 276; a parallel to one side of a triangle cuts off a triangle similar to the first, 278; two triangles are similar if their corresponding sides are proportional, 282; triangles with sides respectively perpendicular or parallel are similar, 283.

5. *Theorems on similar polygons.* Relation of sides and angles, **272, 273**; the perimeters of similar polygons have the same ratio as any two corresponding sides, **296**; in similar polygons any two lines drawn in the same way have the same ratio as any two corresponding sides, **285.**

6. *Theorems on lengths of certain lines of triangles.* The square of the hypotenuse of a right triangle, **290**; the square of side opposite acute angle, **299**; the square of side opposite obtuse angle, **300**; the square of the median, **301**; the product of two sides of a triangle equals the altitude on the third side times the diameter of the circumscribed circle, **302**; the product of two sides of a triangle equals the square of the bisector of their included angle plus the product of the segments of the third side, **303.**

7. *Theorems concerning tangents, secants, and intersecting chords of circle.* Two chords intersecting within a circle, **293**; tangent and secant from an outside point, **294**; two secants from an outside point, **295.**

8. *Constructions.* A fourth proportional to three given lines, **267**; divide a given line into parts proportional to given lines, **286**; a mean proportional between two given lines, **289**; upon a given line construct a polygon, **297.**

9. *Graphical methods.* Drawing to scale, **304**; solution of geometric problems from given data by drawing the figures to scale and measuring the lines or angles required, **305.**

10. *Trigonometric ratios.* Introduction to trigonometry, **308**; definition of the trigonometric functions, **309**; the functions of 60°, **310**; graphical method of computing the functions of an angle, **311**; trigonometric solution of triangles, **312**; extension of the tables, interpolation, **313.**

ARCHIMEDES, THE MASTER SCIENTIST OF ANCIENT TIMES

The mathematical and scientific ability of Archimedes (287–212 B.C.) was, everything considered, second only to that of Newton. His achievements and discoveries were in many fields, — algebra, geometry, astronomy, physics, and engineering. In algebra he derived the relation

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots n^2 = \frac{n}{6}(n+1)(2n+1).$$

He determined very exactly the circumference of a circle when its radius is known, and showed precise knowledge of irrational numbers like $\sqrt{3}$. In geometry he discovered three theorems on the circle, many on the sphere and cylinder, and by methods which anticipated by twenty centuries the calculus of Newton he found the area bounded by a parabola and its chord. In physics he stated clearly the theory of the lever, of parallel forces, and of the equilibrium of floating bodies. This last enabled him to tell King Hiero that the gold and silver crown which the jewelers had made for the king contained all the gold and all the silver given them for that purpose. At the suggestion of the same king he applied his knowledge to build engines of war that threw heavy stones on men and ships. Plutarch tells how effectively these were used by the Syracusans against the Romans. Archimedes computed, using an ingenious notation, the number of grains of sand of a given size required to fill the universe, obtaining a result which in modern symbols would be 10^{63}.

In astronomy Archimedes tried to determine the length of the year more closely than the value then accepted. One of his contemporaries, Aristarchus (about 250 B.C.), saw clearly the plan of the solar system, and by correct measurements imperfectly carried out determined something of its size. Archimedes' statement of the views of Aristarchus is as follows: "He supposes that the stars like the sun remain stationary; that the earth revolves following the circumference of a circle round the sun as a center." It is a significant fact

that the correct theory, here stated eighteen centuries before Copernicus, was rejected by Archimedes. As recently as 1906 there was discovered a letter from Archimedes to his friend in which he disclosed, partially at least, the methods by which he made some of his discoveries.

In the disorder following the capture of Syracuse by the Romans, Archimedes was killed by a soldier. The Roman general Marcellus, who had great respect for Archimedes, regretted this exceedingly. He regarded the slayer as a murderer, and, seeking out the relatives of Archimedes, honored them with marked favors.

The tomb of Archimedes was marked by a monument on which was placed a sphere and a cylinder to commemorate one of his greatest discoveries in geometry. Over a century later, when Syracuse had forgotten even the burial place of the greatest scientist of ancient times, the Roman orator Cicero rediscovered the tomb and identified it by means of the sphere and cylinder, and the inscription upon the monument.

ARCHIMEDES, *killed by a Roman soldier while drawing in the sand* (*287–212* B.C.)

BOOK IV

SURFACE MEASUREMENT, OR AREAS

The sailor whom an exact observation of longitude saves from shipwreck owes his life to a theory developed two thousand years ago by men who had in mind merely the abstract speculations of geometry.

CONDORCET

316. Area of a polygon. A *polygon* has been defined as a closed line. Any polygon incloses a portion of the plane, the area of which is called the area of the polygon. Part of the work of plane geometry is finding the area of the simple plane figures. The area of a triangle, a rectangle, a parallelogram, a trapezoid, etc. means the area of its *interior*. The area of a figure is the number of units of area it contains.

EXERCISES

In the following assume that the unit of area is one of the small squares:

1. Find the areas of Figs. 1 and 2 by counting the small squares.

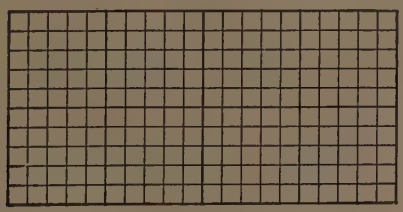


Fig. 1

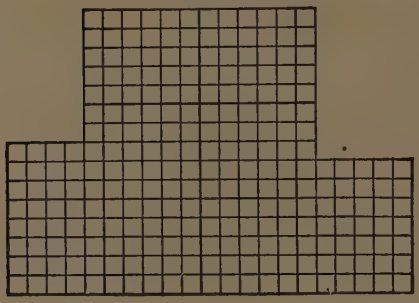


Fig. 2

2. Show by counting the small squares that the approximate areas of Figs. 3 and 4 are 38 and 78 respectively.

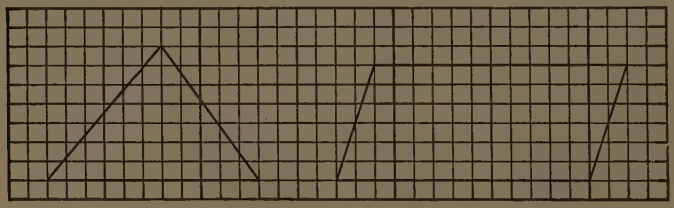


Fig. 3 Fig. 4

3. Find the areas of Figs. 5, 6, and 7 by counting the small squares and estimating as nearly as possible the area of the incomplete small squares near the boundary.

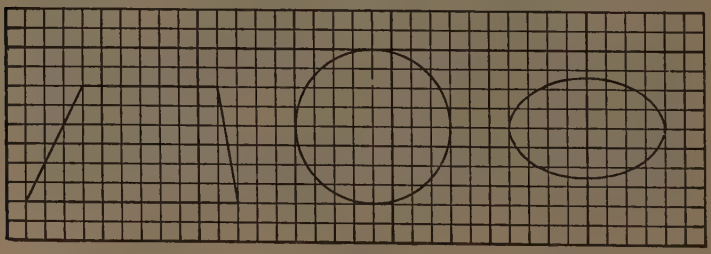
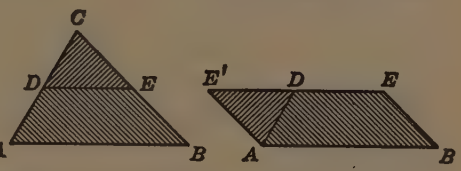


Fig. 5 Fig. 6 Fig. 7

317. Equal plane figures. Two plane figures are *equal* if their areas measured in the same unit are equal.

Two figures having equal areas, whether they are the same shape or not, are sometimes called *equivalent* figures.

318. Congruent plane figures. Two *congruent* figures are superposable and hence *equal*. Two *equal* figures, however, are *not* necessarily *congruent*.



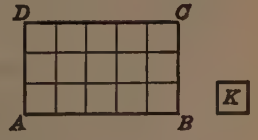
Thus $\triangle ABC$ can be divided into a triangle and a trapezoid, which can be put together to form the parallelogram. The triangle and the parallelogram are equal but are not congruent.

319. Base of a parallelogram. Either one of two adjacent sides of a parallelogram can be considered its *base*.

320. Altitude of a parallelogram. An *altitude* of a parallelogram is a line drawn from any point in one base perpendicular to the opposite side and terminating in it.

A parallelogram has two bases and two altitudes. By "the base and the altitude of a parallelogram" we mean either base and the corresponding altitude. Similarly, "the base and the altitude of a triangle" means any side and the corresponding altitude.

321. Area of a rectangle. The plane figure the measurement of whose surface most clearly illustrates the notion of area is a rectangle whose sides are exact multiples of the unit of length. Thus, rectangle *ABCD* is five units long and three units wide. It can, by parallels to the sides, be divided into fifteen squares each equal to the unit square *K*. We say the *area of the rectangle* is fifteen square units.



When the base or the altitude of the rectangle, or both, and the side of the square have no common unit of measure, the area of the rectangle can still be expressed in terms of K and is equal to the product of the base and altitude. This is a theorem and can be proved. The proof will be omitted, and the assumption which follows will be taken as the basis of the work on areas.

322. Assumption. *The area of a rectangle is the product of its base and altitude.*

It should be noted that the product of two lines always means the product of their numeric measures.

323. ***Corollary 1.*** *Two rectangles having equal bases and altitudes are equal.*

If the base of each is equal to b and the altitude of each is equal to h, then the area of each is bh.

324. ***Corollary 2.*** *Two rectangles are to each other as the product of their bases and their altitudes.*

A_1, the area of the first rectangle, is $h_1 \times b_1$, and A_2, the area of the second, is $h_2 \times b_2$. Hence $A_1 : A_2 = h_1 \times b_1 : h_2 \times b_2$.

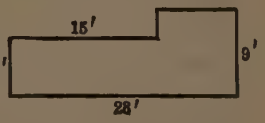
325. ***Corollary 3.*** *The area of a square is the square of one of its sides.*

EXERCISES

1. Can two rectangles be equal if their bases are unequal and their altitudes are unequal?

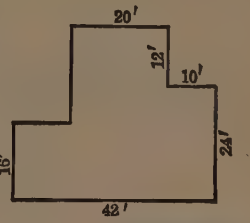
2. A rectangular lot 60 ft. by 140 ft. sold for \$1890. Find the price per square foot.

3. The concrete floor of a porch having the dimensions shown in the adjacent figure cost \$2.88 a square yard. Find the cost of the floor.



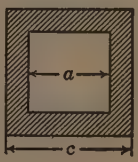
NOTE. The notation ′ and ″ has long stood for minutes and seconds, respectively, of an arc. In building-plans and engineers' drawings, however, it is widely used to designate feet and inches respectively. In many of the numeric exercises this notation will be used.

4. Find the area of the adjacent figure.

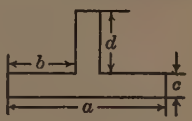


5. A cement sidewalk 2 ft. 3 in. wide and 40 ft. long cost $3.50 per square yard. Find the total cost of the sidewalk.

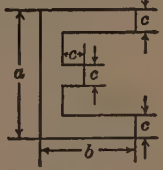
6. The shaded portion between the two squares is of uniform width. Find the shaded area.



7. Show that the area of the adjacent figure is $ac + ad - 2\,bd$. (The length of the right horizontal arm is b.)



8. Show that the area of the adjacent figure is $ac + 2\,bc - c^2$. (The width of the narrow parts of the figure is c.)



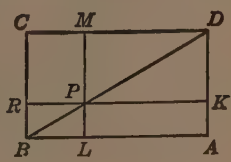
9. The area of a rectangle is 360 sq. ft. One side is 9 ft. Find the diagonal.

10. The area of a rectangle is 1296 sq. ft. The length is six times the altitude. Find the dimensions.

11. The perimeter of a rectangle is 100 ft. and the area is 400 sq. ft. Find the base and the altitude.

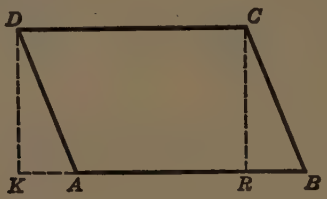
12. Through P, any point on the diagonal BD of $\square\ ABCD$, parallels to the sides are drawn.

Prove $\square\ ALPK = \square\ CMPR$.



Theorem 1

326. *The area of a parallelogram is the product of its base and altitude.*



Given $\square ABCD$, and DK its altitude on the base AB.

To prove that the area of $\square ABCD = AB \cdot DK$.

Method. Construct a rectangle having the same base DC and altitude DK as the given parallelogram. Then $\triangle ADK \cong \triangle BCR$. Then prove the parallelogram equal to the rectangle by subtracting each triangle separately from the whole figure.

Proof

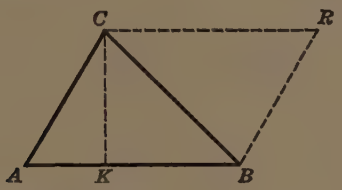
1. Extend BA, and draw the altitudes CR and DK.	1. § 76.
2. $\triangle ADK \cong \triangle BCR$.	2. §§ 126, 128, 109.
3. Taking $\triangle ADK$ from quadrilateral $KBCD$ leaves the $\square ABCD$.	3. § 50, Axiom 2.
4. Taking $\triangle BCR$ from quadrilateral $KBCD$ leaves the $\square KRCD$.	4. § 50, Axiom 2.
5. Then $\square ABCD = \square KRCD$.	5. § 50, Axiom 4.
6. Area of $\square KRCD = DK \cdot DC$.	6. § 322.
7. But $DC = AB$.	7. § 126.
8. $\therefore$ area of $\square ABCD = AB \cdot DK$.	8. Steps 6 and 7.

327. ***Corollary.*** *Any two parallelograms are to each other as the product of their bases and their altitudes.*

For the area A_1 of the first is a_1b_1, and the area A_2 of the second is a_2b_2. Therefore $A_1 : A_2 = a_1b_1 : a_2b_2$.

Theorem 2

328. *The area of a triangle is one half the product of its base and altitude.*



Given $\triangle ABC$ and its altitude CK.

To prove that the area of $\triangle ABC = \frac{AB \cdot CK}{2}$.

Method. Complete a parallelogram having the same base and altitude as the triangle. Then show that the area of the triangle is half that of the parallelogram.

Proof

1. Through C draw a line $\parallel$ to AB, and through B draw a line $\parallel$ to AC intersecting the first in R.	1. § 99.
2. $ABRC$ is a $\square$.	2. § 121.
3. $\therefore$ area of $\triangle ABC = \frac{1}{2}$ area of $\square ABRC$.	3. § 125.
4. Area of $\square ABRC = AB \cdot CK$.	4. § 326.
5. $\therefore$ area of $\triangle ABC = \frac{AB \cdot CK}{2}$.	5. Steps 3 and 4.

329. *Corollary 1. Any two triangles are to each other as the product of their bases and their altitudes.*

If A_1, the area of the first triangle, is $\frac{a_1 b_1}{2}$, and A_2, the area of the second triangle, is $\frac{a_2 b_2}{2}$, then $A_1 : A_2 = a_1 b_1 : a_2 b_2$.

330. *Corollary 2. Two triangles with equal altitudes are to each other as their bases.*

331. *Corollary 3. Two triangles with equal bases are to each other as their altitudes.*

EXERCISES ON THEOREMS 1 AND 2

1. The base of a parallelogram is 12 ft. and the altitude is 7 ft. Find its area.

2. The base of a parallelogram is 6 ft. 4 in. and the altitude is 4 ft. 8 in. Find the area in square feet.

3. The adjacent sides of a parallelogram are 32 ft. and 50 ft. and their included angle is 30°. Find the area.

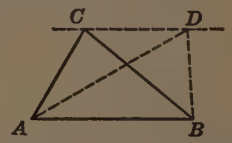
4. The base of a triangle is 8′ and the altitude is 6′. Find the area.

5. The base of a triangle is 10′ 4″ and the altitude is 7′ 9″. Find the area in square feet.

6. Two adjacent sides of a triangle are 12 and 20 respectively. Find the area if the included angle is (*a*) 30°; (*b*) 45°; (*c*) 60°.

7. The hypotenuse of a right triangle is 15′ and one side is 12′. Find the other side and the area of the triangle.

8. In the adjacent figure, CD is parallel to AB. Compare the areas of $\triangle ABC$ and $\triangle ABD$.



9. Find the area of an isosceles right triangle whose hypotenuse is 20′.

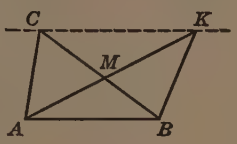
10. Find the area of an equilateral triangle whose side is 18′.

11. Show that the area of any equilateral triangle whose side is s is $\frac{s^2}{4}\sqrt{3}$.

12. Prove that the median of a triangle divides it into two parts of equal area. Are the two parts congruent?

13. The area of an equilateral triangle is $64\sqrt{3}$ sq. ft. Find the side.

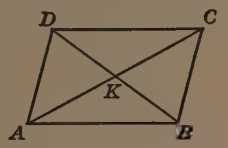
14. The $\triangle ABC$ and the $\triangle ABK$ have a common base, AB, and are on the same side of it. Line CK is parallel to AB, and BC and AK intersect in M.



Prove $\triangle ACM = \triangle BKM$.

15. KR is parallel to AB of $\triangle ABC$ and cuts AC in K and BC in R. AR and BK are drawn.

Prove $\triangle ARC = \triangle BKC$.



16. The diagonals of $\square ABCD$ intersect at K.

Prove $\triangle AKB = \triangle AKD$.

17. The area of a rhombus is half the product of its diagonals. Prove.

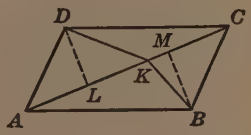
HINT. At what angle do the diagonals of a rhombus meet?

18. If the three altitudes of a triangle are equal, the triangle is equilateral. Prove.

19. K is any point on the diagonal AC of $\square ABCD$.

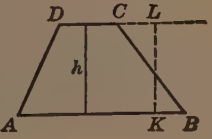
Prove $\triangle BKC = \triangle DKC$.

HINT. Prove $DL = BM$.



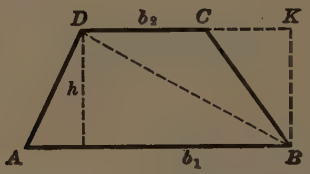
332. Altitude of a trapezoid. The *altitude* of a trapezoid is the perpendicular drawn from any point in one base to the other, produced if necessary.

Thus in the trapezoid $ABCD$ either h or LK is the altitude of the trapezoid.



Theorem 3

333. *The area of a trapezoid is one half the product of its altitude and the sum of its bases.*



Given the trapezoid $ABCD$, in which b_1 and b_2 are the bases and h is the altitude.

To prove that the area of trapezoid $ABCD = \frac{h(b_1 + b_2)}{2}$.

Method. Draw one diagonal. Then write the expression for the area of each triangle thus formed and add the two results.

Proof

1. Draw the diagonal BD and the altitude BK.	1. § 51, Postulate 1; § 76.
2. $BK = h$.	2. § 129.
3. $\triangle ABD = \frac{h \times b_1}{2}$.	3. § 328.
4. $\triangle DCB = \frac{BK \times b_2}{2} = \frac{h \times b_2}{2}$.	4. § 328 and step 2.
5. But the trapezoid $ABCD = \triangle ABD + \triangle DCB$.	5. § 50, Axiom 2.
6. $\therefore ABCD = \frac{h \times b_1}{2} + \frac{h \times b_2}{2} = \frac{h(b_1 + b_2)}{2}$.	6. Steps 3, 4, and 5.

QUERY 1. Is the area of a parallelogram also equal to one half the product of its altitude and the sum of its bases?

QUERY 2. Could a triangle be thought of as a trapezoid with its upper base zero in length?

EXERCISES

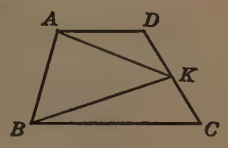
1. Find the area of a trapezoid whose bases are 12′ and 16′ and whose altitude is 7′.

2. The bases of a trapezoid are 6′ 8″ and 8′ 4″ respectively and the altitude is 7′ 3″. Find the area in square feet.

3. Prove that the area of a trapezoid is equal to the product of its altitude and the line joining the mid-points of the nonparallel sides.

4. Prove that the line joining the mid-points of the bases of a trapezoid divides it into two equal trapezoids.

5. AB and CD are the nonparallel sides of a trapezoid and K is the mid-point of CD. Draw AK and BK and prove that $\triangle ABK$ is half the area of the trapezoid.

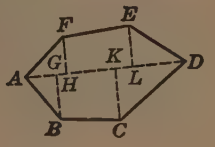


334. Finding the area of any polygon. The area of any polygon can be found in several ways provided the proper dimensions are known.

(*a*) The polygon may be divided into triangles. If their altitudes and bases are known, the area of each and the total area can be computed.

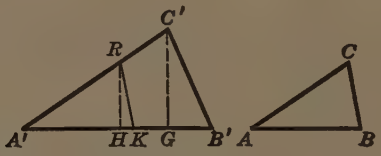
(*b*) The polygon may be divided into triangles and trapezoids. If the proper altitudes and bases are known or can be determined, the area of each figure and the total area can then be computed.

In dividing a polygon it is desirable, as in the adjacent figure, to draw the longest diagonal and then draw perpendiculars to it from the vertices.



Theorem 4

335. *If two triangles have an angle of one equal to an angle of the other, their areas are to each other as the product of the sides including the angle of the first is to the product of the sides including the angle of the second.*



Given $\triangle ABC$ and $A'B'C'$, **in which** $\angle A = \angle A'$.

To prove that $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{AB \cdot AC}{A'B' \cdot A'C'}$.

Method. Place $\triangle ABC$ on $\triangle A'B'C'$ so that the equal angles coincide, and draw the altitudes RH and $C'G$. Write the expression for the area of the first triangle divided by the second in terms of their bases and altitudes. Then show that the ratio of their altitudes may be replaced by the ratio $AC : A'C'$.

Proof

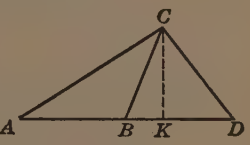
1. Place $\triangle ABC$ in the position $A'RK$ and draw altitudes RH and $C'G$.	1. § 51, Postulate 7. By hypothesis, $\angle A = \angle A'$.
2. Then $\triangle ABC \cong \triangle A'KR$.	2. Step 1.
3. $\dfrac{\triangle A'KR}{\triangle A'B'C'} = \dfrac{\frac{1}{2}(A'K \cdot RH)}{\frac{1}{2}(A'B' \cdot C'G)} = \dfrac{A'K}{A'B'} \cdot \dfrac{RH}{C'G}$.	3. § 328.
4. $\triangle A'HR \sim \triangle A'GC'$.	4. § 277.
5. $\dfrac{RH}{C'G} = \dfrac{A'R}{A'C'}$.	5. § 284.

6. Then $\frac{\triangle A'KR}{\triangle A'B'C'} = \frac{A'K \cdot A'R}{A'B' \cdot A'C'}$. 6. Substituting $\frac{A'R}{A'C'}$ for its equal, $\frac{RH}{C'G}$, in step 3.

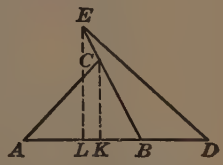
7. $\therefore \frac{\triangle ABC}{\triangle A'B'C'} = \frac{AB \cdot AC}{A'B' \cdot A'C'}$. 7. Steps 2 and 6.

EXERCISES

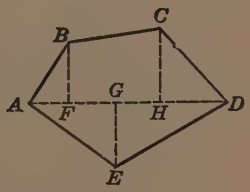
1. Two triangles are equal if two sides of one are equal respectively to two sides of the other and the included angles are supplementary. Prove.



2. If two triangles have an angle of one supplementary to an angle of the other, they are to each other as the product of the sides including the angle of the first is to the product of the sides including the angle of the second. Prove.



3. In the figure of § 334 suppose that the diagonal $AD = 40'$, and that $AG = GH = HK = KL = LD$. If $FH = 12'$, $LE = 18'$, $CK = 10'$, and $BG = 14'$, find the area of the polygon $ABCDEF$.



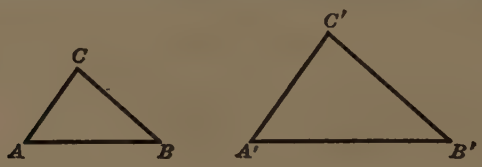
4. In the pentagon $ABCDE$ the diagonal $AD = 44'$, $AB = 20'$, $BF = 12'$, $CH = 18'$, $EG = 16'$, and $FH = 24'$. Find the area of the pentagon.

5. In $\triangle ABC$, K is the mid-point of AC and L is the mid-point of BC. The $\square KLMR$ is drawn so that RM is on AB. Prove $\square KLMR = \frac{1}{2} \triangle ABC$.

6. Through the vertices of a quadrilateral parallels are drawn to the diagonals. Prove that the area of the quadrilateral thus formed is twice that of the first.

Theorem 5

336. *The areas of two similar triangles are to each other as the squares of any two corresponding sides.*



Given the similar $\triangle ABC$ **and** $A'B'C'$.

To prove that $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2} = \dfrac{\overline{AC}^2}{\overline{A'C'}^2} = \dfrac{\overline{BC}^2}{\overline{B'C'}^2}$.

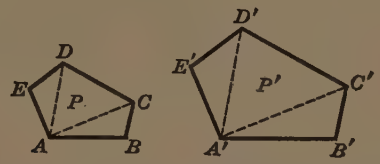
Method. Show that the areas of the two triangles are to each other as the product of the sides including any two equal angles and that these are the products of two ratios of corresponding sides. Prove that this product of ratios equals the ratio of the squares of two corresponding sides. Extend this proof to all three sides.

Proof

1. $\angle A = \angle A'$.	1. § 272.
2. $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{AB \cdot AC}{A'B' \cdot A'C'} = \dfrac{AB}{A'B'} \cdot \dfrac{AC}{A'C'}$.	2. § 335.
3. But $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$.	3. § 272.
4. $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{AB}{A'B'} \cdot \dfrac{AB}{A'B'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$.	4. Steps 2 and 3 and § 50, Axiom 7.
5. $\dfrac{\overline{AB}^2}{\overline{A'B'}^2} = \dfrac{\overline{BC}^2}{\overline{B'C'}^2} = \dfrac{\overline{AC}^2}{\overline{A'C'}^2}$.	5. Step 3 and § 50, Axiom 5.
6. $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2} = \dfrac{\overline{AC}^2}{\overline{A'C'}^2} = \dfrac{\overline{BC}^2}{\overline{B'C'}^2}$.	6. § 50, Axiom 1.

Theorem 6

337. *The areas of two similar convex polygons are to each other as the squares of any two corresponding sides.*



Given the similar convex polygons P and P', in which the sides AB, BC, etc. correspond respectively to the sides $A'B'$, $B'C'$, etc.

To prove that $\frac{P}{P'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2}$, *etc.*

Method. Draw diagonals from two corresponding vertices, and prove that the area of each polygon is made up of the same number of triangles similar in pairs. Then prove that the areas of two corresponding triangles have the same ratio as the squares of any two corresponding sides. Use § 263, 7 and prove that the ratio of the sums of the two sets of triangles is the same as the squares of corresponding sides.

Proof

1. From A and A' draw all the diagonals possible.	1. § 51, Postulate 1.
2. Then $\triangle ABC \sim \triangle A'B'C'$, $\triangle ACD \sim \triangle A'C'D'$, etc.	2. §§ 272, 281.
3. $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}$, etc.	3. § 272.
4. $\therefore \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2} = \frac{\overline{CD}^2}{\overline{C'D'}^2}$, etc.	4. § 50, Axiom 5.

5. $\frac{\triangle ABC}{\triangle A'B'C'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}$, $\frac{\triangle ACD}{\triangle A'C'D'} = \frac{\overline{CD}^2}{\overline{C'D'}^2}$, $\frac{\triangle ADE}{\triangle A'D'E'} = \frac{\overline{DE}^2}{\overline{D'E'}^2}$. 5. § 336.

6. $\frac{\triangle ABC}{\triangle A'B'C'} = \frac{\triangle ACD}{\triangle A'C'D'} = \frac{\triangle ADE}{\triangle A'D'E'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2}$, etc. 6. Steps 4 and 5.

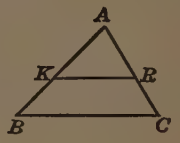
7. $\therefore \frac{\triangle ABC + \triangle ACD + \triangle ADE}{\triangle A'B'C' + \triangle A'C'D' + \triangle A'D'E'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2}$, etc. 7. § 263, 7.

8. $\therefore \frac{P}{P'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2}$, etc. 8. § 50, Axiom 2.

EXERCISES ON THEOREMS 5 AND 6

1. One side of a triangle is 9 ft. and the area is 54 sq. ft. The corresponding side of a similar triangle is 45 ft. Find its area.

2. In the adjacent figure, AB is 18′, AK is 12′, and KR is parallel to BC. The area of $\triangle ABC$ is 90 sq. ft. Find the area of $\triangle AKR$ and of the trapezoid $BCRK$.

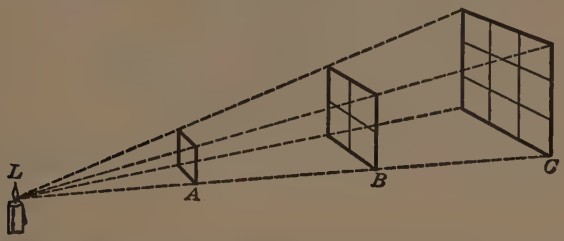


3. One side of a triangle is 100″. Two parallels to another side divide the triangle into three equal parts. Find the three segments of the given side.

4. The corresponding sides of two similar polygons are 15′ and 55′ respectively. The area of the first is 180 sq. ft. Find the area of the second.

5. The areas of two similar polygons are 144 and 625 respectively. One side of the first is 20. Find the corresponding side of the second.

338. The inverse-square law of light. The square B is twice as far from the light L as A, and C is three times as far away as A. The light which falls on A is spread at B over four times as large an area as A, and at C over an area nine times the area of A. That is, the intensity (brightness) of the light from a given source *decreases* as the square of the distance from the source *increases*.



If the air is still, the intensity (loudness) of sound varies inversely with the square of the distance from the source of the sound, precisely as light does.

EXERCISES

1. Compare the brightness of the page of a book when it is 1 ft. from a lamp with that when it is 4 ft. from it.

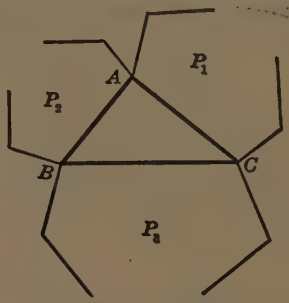
2. If the sun were half as far away as it is, how would its brightness compare with the present brightness?

3. Two men hear a rifle shot. One is 528 ft. distant, the other is a mile away. Compare the intensity of the sound which reaches the two men.

4. A book is held 3 ft. from a lamp. How far must it be held from a second lamp four times as bright as the first in order to be equally illuminated? How far must it be held from the first lamp to be lighted as brightly as it would be when 3 ft. from the second lamp?

Theorem 7

339. *If, on the three sides of a right triangle as the corresponding sides, similar polygons of n sides be constructed, the area of the polygon on the hypotenuse equals the sum of the areas of the polygons on the other two sides.*



Given rt. $\triangle ABC$, and similar polygons P_1, P_2, and P_3 described on AC, AB, and BC respectively as corresponding sides.

To prove that $P_3 = P_1 + P_2$.

Method. Show that the three polygons are to each other as the squares of the three sides forming the right triangle. Then use § 263, 7 and show that the sum of the two smaller polygons is to the larger as the sum of the squares of the smaller sides of the right triangle is to the square of the hypotenuse.

Proof

1. $\frac{P_1}{P_3} = \frac{\overline{AC}^2}{\overline{BC}^2}$, and $\frac{P_2}{P_3} = \frac{\overline{AB}^2}{\overline{BC}^2}$.	1. § 337.
2. $\therefore \frac{P_1 + P_2}{P_3} = \frac{\overline{AC}^2 + \overline{AB}^2}{\overline{BC}^2}$.	2. § 50, Axiom 3.
3. But $\frac{\overline{AC}^2 + \overline{AB}^2}{\overline{BC}^2} = 1$.	3. § 290.
4. $\therefore P_3 = P_1 + P_2$.	4. Steps 2 and 3.

EXERCISES

1. The corresponding sides of two similar polygons are 15′ and 8′ respectively. What is the corresponding side of another similar polygon equal to their sum?

2. The corresponding sides of two similar polygons are 20′ and 29′ respectively. Find the corresponding side of a similar polygon equivalent to their difference.

3. The corresponding sides of two similar polygons are 4′ and 5′ 4″ respectively. Find the corresponding side of a polygon similar to them whose area is nine times their combined areas.

4. A line parallel to AB of $\triangle ABC$ divides it into two parts of equal area. If CB is 10″, find the two parts into which the parallel divides it.

5. K on AB and R on AC of $\triangle ABC$ are points such that the trapezoid $BCRK$ is seven sixteenths of $\triangle ABC$. If AB is 20 centimeters, find AK.

6. Find the area of a trapezoid whose bases are 12″ and 15″ and whose altitude is 10″.

7. The sides of a trapezoid are 25′, 25′, 40′, and 50′ respectively, the last two being the bases. Find the area of the trapezoid.

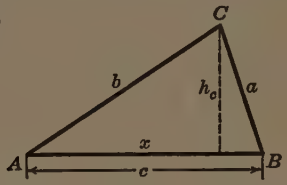
8. The area of a right triangle is 210 sq. in., and one side about the right angle is 1 in. longer than the other. Find the three sides.

9. Find the area and the three altitudes of the right triangle whose sides are 30′, 40′, and 50′.

10. Two sides of a triangle are 30′ and 40′ respectively, and two sides of a second triangle are 28′ and 16′ respectively, the included angles in the two cases being equal. Find the ratio of the area of the triangles.

Problem 1 (*Computation*)

340. *Find the value of an altitude of a triangle in terms of its sides.*



Given $\triangle ABC$, and the altitude h_c on side c.

To express the value of h_c in terms of a, b, and c.

Solution. Let A be an acute angle.

Then $h_c^2 = b^2 - x^2$, (1) § 290

and $a^2 = b^2 + c^2 - 2cx$. (2) § 299

From (2), $x = \dfrac{b^2 + c^2 - a^2}{2c}$. (3)

From (1) and (3),

$$h_c^2 = b^2 - \left(\frac{b^2 + c^2 - a^2}{2c}\right)^2. \quad (4)$$ § 50, Axiom 7

A form better adapted for computation is obtained as follows:

$$h_c^2 = \left(b + \frac{b^2 + c^2 - a^2}{2c}\right)\left(b - \frac{b^2 + c^2 - a^2}{2c}\right) \quad (5)$$

$$= \left(\frac{b^2 + 2bc + c^2 - a^2}{2c}\right)\left(\frac{a^2 - b^2 - c^2 + 2bc}{2c}\right) \quad (6)$$

$$= \frac{[(b+c)^2 - a^2]\,[a^2 - (b-c)^2]}{4c^2} \quad (7)$$

$$= \frac{(b+c+a)(b+c-a)(a+b-c)(a-b+c)}{4c^2}. \quad (8)$$

To simplify (8), let $a+b+c=2s$; then $b+c-a=2(s-a)$, $a+b-c=2(s-c)$, and $a-b+c=2(s-b)$.

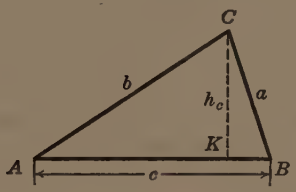
Substituting in (8) gives

$$h_c^2 = \frac{2s(2)(s-a)(2)(s-b)(2)(s-c)}{4c^2}. \quad (9)$$

$$\therefore h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}. \quad (10)$$

Problem 2 (*Computation*)

341. *Find the area of a triangle in terms of its sides.*



Given $\triangle ABC$, in which either $\angle A$ or $\angle B$ is acute.

To compute the area in terms of its sides.

Solution. Area $= \frac{ch_c}{2}$, (1) § 328

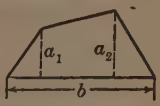
and $h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$. (2) § 340

From (1) and (2), Area $= \sqrt{s(s-a)(s-b)(s-c)}$.

AN ENGINEER AND INVENTOR OF ANCIENT EGYPT

The discovery of the proof of the formula for the area of a triangle in terms of its sides is credited to the Egyptian Hero of Alexandria (about A.D. 50). Hero's proof of the formula for the area of a triangle is very different from the proof given above, being more geometric in character, but it is easily understood. Hero was able to solve the general quadratic equation $ax^2 + bx = c$, a thing Euclid could not do. In his geometry he stated certain formulas which show that he had considerable knowledge of trigonometry also. He found irregular areas by a process corresponding to the use of coördinate paper, as was done in the exercises on page 315. It is remarkable that a mind so able could not derive the correct formula for the area of a quadrilateral. His formula is

$$\frac{a_1 + a_2}{2}\left(\frac{b}{2}\right),$$



in which a_1, a_2, and b are defined as in the adjacent figure.

Hero's activities extended to astronomy, surveying, and physics as well as to mathematics. He devised a surveying instrument equipped with a water level, the whole being in principle much like the modern surveyor's level. Two poles with disks mounted on them were used in connection with his level, just as engineers use them today. Hero wrote on mechanics, treating the lever, the wedge, the screw, the pulley, the wheel and axle, and centers of gravity. Long before the time of Watt, Hero devised an engine with steam as the motive power, which ran on the same principle as that of a rotary lawn-sprinkler.

"The writings of Hero supplied a practical want and for that reason were borrowed extensively by other peoples." — CAJORI

EXERCISES

1. By analogy, from equation (10) of § 340 write the expression for h_a and h_b.

2. Find the longest altitude of a triangle whose sides are 4, 13, and 15.

Solution. Let $a = 4$, $b = 13$, and $c = 15$.

Then $s = 16$.

Substituting in $h_a = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$,

$$h_a = \frac{2}{4}\sqrt{16 \cdot 12 \cdot 3 \cdot 1} = 12.$$

3. Find the altitude on side 6 of the triangle whose other two sides are 25 and 29.

4. The sides of a triangle are 35, 44, and 75. Find the altitude on side 75.

5. If one altitude and the three sides of a triangle are known, what is the easiest way to find the other two altitudes?

6. Find the other two altitudes in Exercises 3 and 4 by the method of Exercise 5.

EXERCISES

1. Compute the area of a triangle whose sides are 7′, 15′, and 20′.

Solution. Here $2s = 7 + 15 + 20 = 42$.

$\therefore s = 21$ and $A = \sqrt{21 \cdot 1 \cdot 6 \cdot 14} = \sqrt{7^2 \cdot 3^2 \cdot 2^2} = 42$.

Find the area of a triangle whose sides are as follows:

2. 11′, 13′, and 20′.

3. 10′, 17′, and 21′.

4. 12′, 17′, and 25′.

5. 11′, 25′, and 30′.

6. 13′, 37′, and 40′.

7. 15′, 28′, and 41′.

8. 21′, 89′, and 100′.

9. 35′, 100′, and 117′.

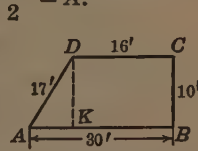
Find the altitudes of the triangles whose sides are as follows:

10. 9′, 10′, and 17′.

11. 20′, 37′, and 51′.

HINT. The area A of the triangle may be found by the formula of § 341. Then if x, y, and z are the altitudes on the sides 9, 10, and 17 respectively, $\frac{9x}{2} = A$, $\frac{10y}{2} = A$, and $\frac{17z}{2} = A$.

12. Find the area of a trapezoid whose sides are 10′, 17′, 16′, and 30′ respectively.



HINT. Draw $DK \parallel CB$, and find the altitude on side AK of $\triangle ADK$.

13. Find the diameter of the circle circumscribing the triangle whose sides are 13, 20, and 21.

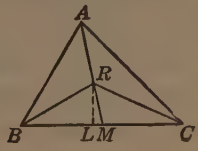
HINT. See § 302.

MISCELLANEOUS EXERCISES

1. Does the bisector of an angle of a triangle divide it into two equivalent parts? Prove.

2. R is any point on the median AM of $\triangle ABC$. Draw BR and CR.

Prove $\triangle ABR$ and ACR equivalent.



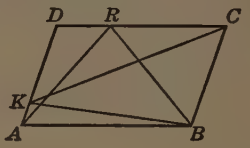
3. The mid-points of two sides of a triangle are joined to any point in the third side.

Prove that the area of the quadrilateral thus formed is one half the area of the triangle.

4. Given any point K within $\square ABCD$, draw lines from K to the four vertices.

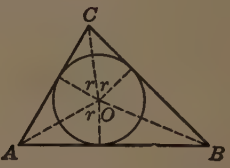
Prove that the sum of the areas of $\triangle KAB$ and KCD is one half the area of the parallelogram.

5. Given K any point on the side AD and R any point on the side DC of $\square ABCD$, draw lines AR, BR, KB, and KC.



Prove that $\triangle BKC$ and ARB are equivalent.

6. The area of a triangle is one half the product of its perimeter and the radius of its inscribed circle.



7. The sum of the perpendiculars from any point within an equilateral triangle to the sides is equal to the altitude of the triangle.

HINT. Use Theorem 2, § 328.

8. Prove that the radius of the inscribed circle of a triangle whose sides are a, b, and c respectively is

$$\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

9. $ABCD$ is a quadrilateral divided into equivalent parts by KR, the point K being on AB and the point R on CD. The point L is on KB. A parallel to LR through K cuts CD (not CD produced) in H. Prove that LH also bisects the quadrilateral.

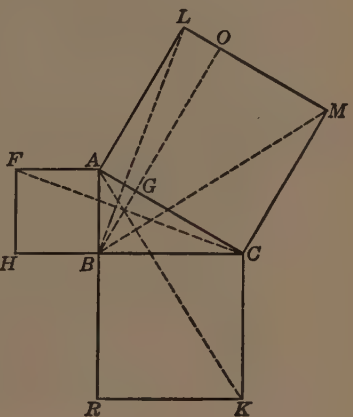
10. Parallels are drawn through the vertices of the $\triangle ABC$ cutting the sides or the sides produced in K, R, and L respectively. Draw KR, RL, and LK.

Prove that the area of $\triangle KRL$ is twice that of $\triangle ABC$.

11. R and K are any two points on the sides CD and BC respectively of the $\square ABCD$. A parallel to AK through D cuts AR produced in L. A parallel to AL through K cuts DL produced in M.

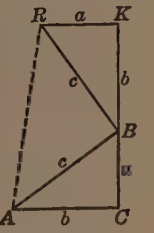
Prove $AKML = ABCD$.

12. Prove that the area of the square on the hypotenuse of a right triangle equals the sum of the areas of the squares on the other two sides.



HINT. In the adjacent figure,
$\triangle ACK \cong \triangle BCM$.
$\therefore \square OMCG = \square BCKR$.
Similarly,
$\triangle ABL \cong \triangle AFC$.
$\therefore \square ALOG = \square ABHF$.
Etc.

13. Prove that $a^2 + b^2 = c^2$ in the right triangle ABC, using the adjacent figure, which was devised by President Garfield in an original proof of the Pythagorean Theorem.



Solution. $ACKR = \frac{a+b}{2}(a+b) = \frac{(a+b)^2}{2}$.

$\triangle ABC = \frac{ab}{2}$, $\triangle RBK = \frac{ab}{2}$, and $\triangle ABR = \frac{c^2}{2}$.

$$\therefore \frac{(a+b)^2}{2} = \frac{ab}{2} + \frac{ab}{2} + \frac{c^2}{2},$$

or $$a^2 + b^2 = c^2.$$

PYTHAGORAS AND HIS FAMOUS THEOREM

Egyptian engineers before Pythagoras (572–501 B.C.) and modern workmen today use the special 3–4–5 right triangle to lay out a right angle. The first general proof, however, that the square on the hypotenuse of any right triangle equals the sum of the squares on the other two sides is due to Pythagoras, who was the ablest pupil of Thales. Many proofs of this theorem now exist, but which one of them is his remains unknown. It is probable that he proved also that the sum of the angles of a triangle is two right angles. He perceived that a right triangle having two sides each equal to 1 had a hypotenuse equal to $\sqrt{2}$. Thus to the integers and fractions he added a third type of number, the irrational number. He or his school knew that either six equilateral triangles, four squares, or three regular hexagons filled up the space about a point in the plane. Pythagoras took the first scientific steps in music in connection with the length of vibrating strings. It is now well known that strings whose lengths are 6, 4, 3 will give out a fundamental, the fifth above it, and the octave, as C, G, and C'. This fundamental relation was discovered by Pythagoras. In astronomy he taught that the earth was round. He was then a moralist and philosopher and tried to construct a mathematical basis for his teachings in those lines. In all his work his ideal was high, knowledge, not wealth, being the object of his study.

EXERCISES ON MENSURATION

1. Find the area in square inches of a triangle whose base is 7 ft. and whose altitude is $4\frac{2}{3}$ ft.

2. Find the area in square feet of a parallelogram whose base is $3\frac{3}{4}$ ft. and whose altitude is 32 in.

3. Find the area in square yards of a rectangle whose base is $22\frac{1}{2}$ ft. and whose altitude is 64 in.

4. A field is 33 rd. wide and 120 rd. long. How many acres are there in it?

5. The base of a triangle is 86 rd. and its altitude is 35 rd. Find its area in acres.

6. The base of a triangle is 12 ft. and its area is 8 sq. yd. Find the altitude.

7. A piece of ground in the form of a right triangle was sold for $337.50 at $150 per acre. One side was 40 rd. Find the hypotenuse and the other side.

8. How many acres are there in a rhombus in which each side is 825 ft. and the altitude is 528 ft.?

9. A piece of sidewalk is 66 ft. 8 in. long and 4 ft. 6 in. wide. What did it cost at $2.98 a square yard?

10. A rectangular field contains 11 A. and is 32 rd. wide. How much will it cost to fence it at 32 cents a rod?

11. Two parallel sides of a field are 74 rd. and 49 rd. long and the distance between them is 20 rd. Find the area.

12. A piece of ground is 90 ft. wide at one end and 75 ft. at the other. Its area is 396 sq. yd. Find the length.

13. A city lot is 75 ft. wide in front, 55 ft. at the rear, and 150 ft. deep. It sells for $1560. Find the price per square foot.

14. One diagonal of a quadrilateral is 242 ft. and the perpendiculars to the diagonal from the other vertices are 112 ft. and 140 ft. Find its area in acres.

15. How many bricks 9 in. long and $4\frac{1}{2}$ in. wide are needed to pave a road 2 mi. long and 22 ft. wide?

16. The hypotenuse of a right triangle is 15 in. and one of the other sides is 3 in. longer than the third side. Find the three sides.

17. One angle of a triangle is 135°, and the sides including it are 72 in. and 56 in. respectively. Find the third side of the triangle. (Use §§ 290, 291.)

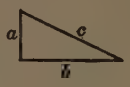
Problem 3 (*Construction*)

342. *Construct a square equal to the sum of two given squares.*

Given the two squares with sides a and b respectively.

Required to construct a square equal to their sum.

Construction. Construct a right triangle with a and b the sides about the right angle. Then the hypotenuse c is the side of the required square.



Proof

1. $a^2 + b^2 = c^2$.	1. § 290.

EXERCISES

1. Construct a square equal to the difference of two given squares.

2. Construct a square equal to the sum of three given squares.

3. Show how to construct a square equal to the sum of four given squares.

4. Construct a line equal to $a\sqrt{2}$.

HINT. Construct a right triangle with two sides equal to a.

5. Construct a line equal to $a\sqrt{3}$.

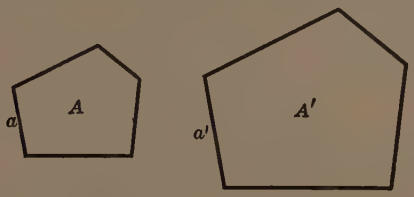
HINT. Construct a right triangle with one side equal to a and the hypotenuse equal to $2a$.

6. Construct a line equal to $a\sqrt{5}$.

HINT. Construct a right triangle with two sides about the right angle equal to a and $2a$ respectively.

Problem 4 (*Construction*)

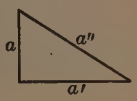
343. *Construct a polygon equal to the sum of two given similar polygons and similar to them.*



Given two similar polygons A and A'.

Required to construct a polygon similar to A and A' and with an area equal to the combined areas of A and A'.

Construction. Construct a right triangle with the corresponding sides a and a' the sides about the right angle. The hypotenuse a'' is the side of the required polygon equal in area to $A + A'$. Then construct on a'' a polygon A'' similar to A, as in § 297.



Proof

1. A'' is similar to A and A'.	1. By construction.
2. $a^2 + (a')^2 = (a'')^2$.	2. § 290.
3. $\therefore A + A' = A''$.	3. § 339.

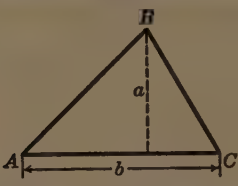
EXERCISES

1. Construct an equilateral triangle whose area is equal to the sum of the areas of two given equilateral triangles.

2. The sides of a triangle are 5, 6, and 7. Another triangle similar to it has a side 10 corresponding to the side 5. Construct a triangle whose area equals and is similar to the area of the two triangles.

Problem 5 (*Construction*)

344. *Construct a square equal to a given triangle.*



Given $\triangle$ *ABC* **with base** *b* **and altitude** *a*.

Required to construct a square equivalent to $\triangle$ *ABC*.

Method. Let x be the side of the required square. Then its area is x^2. The area of $\triangle ABC$ is $\frac{ab}{2}$. Hence $x^2 = \frac{ab}{2}$. Then $\frac{a}{2} : x = x : b$.

Construction. Construct x a mean proportional between $\frac{a}{2}$ and b, as in § 289. Then construct a square whose side is x.

Proof

1. The area of the square is x^2.	1. § 325.
2. The area of $\triangle ABC$ is $\frac{ab}{2}$.	2. § 328.
3. $x^2 = \frac{ab}{2}$.	3. By construction.

345. Use of an equation in construction problems. Most of the construction problems of Book IV require the construction of a figure equal to a given one or to a certain definite part of a given one. Such problems can be solved most readily as follows:

1. Express as an equation the stated or implied relation. Then represent the unknown line by x and express the area of the required figure in terms of x and the given parts.

2. Express the area of the given figure in terms of its dimensions.

3. Use the conditions stated in the problem and state an equation, as was done in § 344. An inspection of this equation, when put into the form of a proportion, will show whether a mean proportional or a fourth proportional is required to solve the given problem.

Thus the solution of a construction problem involving areas can very often be made to depend on either a mean proportional or a fourth proportional or on both.

EXERCISES

1. Construct a square whose area is equal to the area of a given rectangle.

HINT. $s^2 = ab$. Hence $a : s = s : b$. Etc.

2. Construct a square equivalent to a given parallelogram.

3. Construct a square equivalent to a given trapezoid.

4. Construct a square which is three fifths of a given rectangle.

5. Construct a square equal to a given rhombus.

6. Construct a triangle equal to a given square.

7. Transform a triangle into an isosceles triangle having the same base and area.

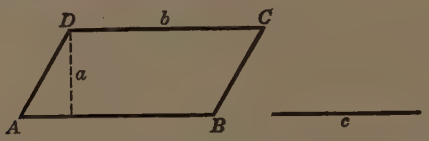
8. Construct a square having twice the area of a given square.

9. Construct a square having three times the area of a given square.

10. Transform a triangle into an equal triangle having the same base and a given angle adjacent to the base.

Problem 6 (*Construction*)

346. *On a given line as the base, construct a triangle equal to a given parallelogram.*



Given c, the base of the required triangle, and $\square ABCD$, whose altitude is a and whose base is b.

Required to construct a triangle equivalent to ABCD on c as the base.

Method. Let x be the unknown altitude of the required triangle. Then its area is $\frac{cx}{2}$. That of the parallelogram is ab. Therefore $\frac{cx}{2} = ab$. Hence $\frac{c}{2} : a = b : x$.

Construction. Construct a fourth proportional to $\frac{c}{2}$, a, and b, as in § 267. Then construct a triangle whose base is c and whose altitude is x.

Proof

1. The area of the $\triangle$ is $\frac{cx}{2}$.	1. § 328.
2. The area of the given $\square$ is ab.	2. § 326.
3. $\frac{cx}{2} = ab$.	3. By construction.

QUERY 1. How many triangles may be constructed which satisfy the conditions of § 346?

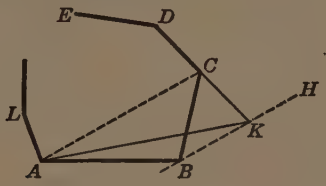
QUERY 2. Is more than one solution possible for the problem of § 344? of § 343? of § 342?

QUERY 3. How many parallelograms may be constructed equivalent to a given square?

QUERY 4. State how many solutions are possible for the following: On a given base construct a parallelogram equivalent to a given square.

Problem 7 (*Construction*)

347. *Transform a polygon of n sides into an equivalent polygon of n − 1 sides.*



Given the polygon *ABCDE* ··· having *n* sides.

Required to transform the polygon ABCDE ··· into an equivalent polygon having n − 1 sides.

Construction. Draw any diagonal which joins two alternate vertices, as *AC*. Through *B* draw *BH* ∥ to *AC*. Extend *DC* (or *LA*) to meet *BH* at *K*. Draw *AK*. Then *AKDE* ··· is equivalent to *ABCDE* ··· and has *n* − 1 sides.

Proof

1. $\triangle ACB = \triangle ACK$.	1. Same base, *AC*, and equal altitudes.
2. Figure $ACDE\cdots + \triangle ACK$ = figure $ACDE\cdots + \triangle ACB$.	2. § 50, Axiom 3.
3. Figure $AKDE\cdots$ has $n-1$ sides and is the required polygon.	3. Sides *DC*, *CB*, and *AB* are replaced by *DK* and *AK*.

EXERCISES

1. On a given base construct a triangle equivalent to a given square.

2. On a given base construct a triangle equivalent to a given triangle.

3. On a given base construct a triangle equivalent to the sum of two given triangles.

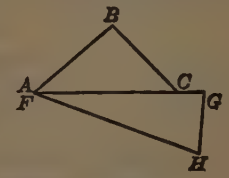
4. Using the method of Problem 7, transform a quadrilateral into an equal triangle.

5. Transform a pentagon into an equal triangle.

6. Transform a hexagon into an equal triangle.

7. Explain how a polygon of n sides may be transformed into an equal triangle.

8. Construct a triangle equal to the sum of two given triangles.



HINT. Place the triangles adjacent as indicated and then use Problem 7.

9. Construct a trapezoid equal to a given square.

10. Construct a rectangle on a given base equal to a given rectangle.

11. Construct a triangle equal to a given seven-sided polygon.

12. Divide a triangle into three equal parts by lines from one vertex.

13. Divide a parallelogram into three equal parts by lines from one vertex.

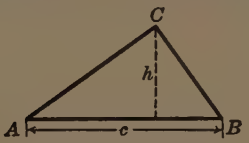
14. Divide a parallelogram into two equal parts by a line through a given point in one of the bases.

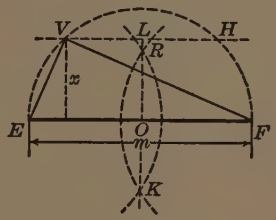
15. Construct a triangle equal to the difference of two similar triangles and similar to them.

348. Constructions involving areas and loci. Many problems of construction involve equal areas and loci. A construction problem may involve the condition that the required figure have an area equal to a given one. In addition to this there may be a condition involving a locus, as described in Book II. The following example illustrates both these points.

ILLUSTRATIVE CONSTRUCTION

Given the hypotenuse, construct a right triangle equivalent to a given triangle.





Given $\triangle ABC$ **and the line** EF.

Required to construct a right triangle having EF ***as the hypotenuse and equivalent to*** $\triangle ABC$.

Method. Let c and h be the base and the altitude of the given triangle ABC, and let x be the unknown altitude upon the given hypotenuse m, or EF.

Then $\frac{ch}{2} = \frac{mx}{2}$. (1) § 328

Hence $m : c = h : x$. (2) § 263, 2

Construction. Construct x, a fourth proportional to m, c, and h, as in § 267. Then construct KR, the perpendicular bisector of EF at O. Draw semicircle EF. On OR lay off $OL = x$. Construct $LH \perp$ to OL, cutting the semicircle in V. Draw EV and FV. Then EVF is the required triangle.

Proof

1. $\angle EVF = 90°$.	1. § 230.
2. Altitude of $\triangle EVF = x$.	2. By construction.
3. $\therefore \triangle EVF = \triangle ABC$.	3. Equations (1) and (2).
4. $\therefore \triangle EVF$ is the required $\triangle$.	4. Steps 1 to 3.

The required triangle must have the given hypotenuse as its base, and its altitude must be x.

Hence the vertex of the right angle must be on the semicircle EVF and also on a line parallel to the hypotenuse at a distance from it equal to x. Hence the vertex must be at one of the points of intersection of these two loci.

QUERY. What is the relation between m, c, and h which makes a solution possible?

PROBLEMS OF CONSTRUCTION INVOLVING AREAS

Solve the following problems, and state the conditions under which the problem might have no solution, one solution, or more than one solution.

1. Construct a triangle equivalent to a given trapezoid.

2. Divide a triangle into five equivalent parts by lines drawn from one vertex.

3. Through a given point within a parallelogram draw a line dividing it into two equivalent parts.

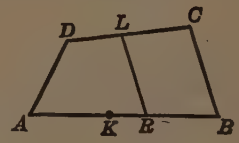
4. Divide a parallelogram into four equivalent parts by lines through one vertex.

5. Given two sides, construct a triangle equivalent to a given triangle.

6. Divide a parallelogram into two equivalent parts by a line perpendicular to one side.

7. Given one diagonal, construct a rhombus equivalent to a given parallelogram.

8. RL in the adjacent figure divides quadrilateral $ABCD$ into two equivalent parts. Draw through K a line which will also divide $ABCD$ into two equivalent parts.



HINT. See Exercise 9, page 338.

9. Construct an equilateral triangle whose area is twice that of a given equilateral triangle.

REVIEW QUESTIONS

1. What is the unit of area?

2. What is the meaning of "area"?

3. What is the difference between congruent and equal figures?

4. Are congruent figures equal? Are equal figures congruent?

5. Are two equal circles necessarily congruent? two equal squares? two equal equilateral triangles? two equal isosceles triangles? two equal rectangles?

6. What is the formula for the area of (*a*) a square? (*b*) a rectangle? (*c*) a triangle? (*d*) a trapezoid? (*e*) a triangle in terms of its sides? (*f*) a right triangle?

7. What lines should be measured or known in order to find the area of (*a*) a parallelogram? (*b*) a triangle? (*c*) a trapezoid?

8. In what case is one side of a triangle an altitude? one side of a parallelogram?

9. State the formula for (*a*) the altitude of an equilateral triangle in terms of its side; (*b*) its area in terms of its side.

10. What is the ratio of the area of a square to that of an equilateral triangle if the side of each is s? Is this ratio commensurable?

11. Is a triangle divided into two equal triangles by a median? by a bisector of one of its angles? by an altitude?

12. The perimeter of one equilateral triangle is four times that of a second triangle. Compare their areas.

13. One equilateral triangle has nine times the area of a second triangle. Compare their perimeters.

14. The altitude of an isosceles triangle is doubled. What is the change in the area?

15. The altitude of an equilateral triangle is doubled. What is the change in the area?

16. When is the projection of a line on another line equal to itself? When is it a point?

17. Draw $\triangle ABC$ and its altitudes, AK, BL, and CM. What is the projection of (*a*) AB on BC? (*b*) AB on AC? (*c*) BC on AB? (*d*) BC on AC? (*e*) AC on BC? (*f*) AC on AB?

18. The projection of side 5 on side 12 of a parallelogram is 3. Find the area of the parallelogram.

19. Two corresponding sides of two similar polygons are 10 and 20 respectively. Find (*a*) the ratio of their perimeters; (*b*) the ratio of their areas.

20. Two sides of a triangle, a and b, are of fixed length, and the included angle is variable. If the angle starts at 10° and increases, does the area increase? What is the greatest area the triangle can have?

21. Two sides of a triangle are trebled and the third side is left unchanged. What is the change in the area?

22. AB is the diameter of a circle of radius r and ABC is an inscribed triangle. What is the greatest area $\triangle ABC$ can have?

23. Complete the following sentence:

Two similar areas are to each other as ____ of any two corresponding lines.

24. Draw an irregular pentagon, and show what lines must be measured in order to compute its area.

25. The projection of side 10 on side 18 of a triangle is 6. What is the area of the triangle?

Summary

(Black figures refer to sections)

349. Résumé of Book IV. Book IV deals fundamentally with the areas of some of the simpler geometric figures and the relation existing between areas of similar figures.

1. *Definitions.* Area of a polygon, **316**; area of a rectangle, **321**; equal figures, **317**; equivalent figures, **317**; distinction between congruent figures and equal figures, **318**; base of a parallelogram, **319**; altitude of a parallelogram, **320**; altitude of a trapezoid, **332**.

2. *Theorems on areas.* Area of a rectangle, **321, 322**; area of a square, **325**; area of a parallelogram, **326**; area of a triangle in terms of its base and altitude, **328**; area of a trapezoid, **333**; relation of the areas of any two triangles, **329, 335**; altitude of a triangle in terms of its three sides, **340**; area of a triangle in terms of its sides, **341**.

3. *Areas of similar figures.* Two similar triangles, **336**; two similar convex polygons, **337**; inverse-square law of light and sound, **338**.

4. *Theorem of Pythagoras extended to similar polygons.* Polygon on hypotenuse equals sum of similar polygons on the other two sides of the right triangle, **339**.

5. *Construction problems.* Square equal to the sum of two given squares, **342**; polygon equal to the sum of two similar polygons, **343**; square equal to a given triangle, **344**; triangle on a given base equal to a given parallelogram, **346**; use of an equation in construction problems, **345**; transformation of a polygon of n sides into an equivalent one of $n - 1$ sides, **347**; use of loci in problems of construction, **348**.

The Romans used the semicircular arch in almost all their buildings, because it is the strongest and the simplest stone arch

BOOK V

REGULAR POLYGONS AND THE CIRCLE

Our science in contrast with others is not founded on a single period of human history but has accompanied the development of culture throughout the ages. Mathematics is as much interwoven with Greek culture as with the most modern problems of engineering. She not only lends a hand to the progressive natural sciences but participates at the same time in the abstract investigations of logicians and philosophers.

KLEIN

350. Regular polygon. A *regular polygon* is one that is equiangular and equilateral (see §§ 152, 153, 154).

351. Uses of regular polygons. One important object of this book is the measurement of the circle. The method of measurement consists in comparing the length and area of the circle with the perimeter and area respectively of the regular inscribed and regular circumscribed polygons.

QUERY 1. Is a polygon all of whose angles are equal necessarily regular? Illustrate.

QUERY 2. Is a polygon all of whose sides are equal necessarily regular? Illustrate.

QUERY 3. In the preceding queries substitute the word "triangle" for "polygon," and answer.

QUERY 4. What is the name of a regular polygon of three sides? of four sides? of five sides? of six sides? of eight sides? of ten sides?

QUERY 5. How many degrees are there in each angle of a regular polygon of three sides? of four sides? of five sides?

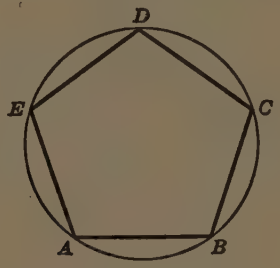
QUERY 6. What is the formula used in finding one angle of a regular polygon of n sides?

QUERY 7. When is a polygon inscribed in a circle?

QUERY 8. When is a polygon circumscribed about a circle?

Theorem 1

352. *An equilateral polygon inscribed in a circle is regular.*



Given the equilateral polygon *ABCDE* inscribed in a circle.

To prove that ABCDE is regular.

Method. Prove that the equal sides have equal arcs. Then show that the angles of the polygon are measured by halves of equal arcs, and the theorem is proved.

Proof

1. Chords AB, BC, CD, etc. are equal.	1. By hypothesis.
2. Arcs AB, BC, CD, etc. are equal.	2. § 186.
3. Arcs ABC, BCD, CDE, etc. are equal.	3. § 50, Axiom 3.
4. $360° -$ arc $EAB = 360° -$ arc ABC.	4. § 50, Axiom 4.
5. $\therefore \angle A = \angle B$.	5. Step 4 and § 229.
6. Similarly, $\angle B = \angle C = \angle D = \angle E$.	6. Steps 1 to 4.
7. $\therefore$ $ABCDE$ is regular.	7. § 350.

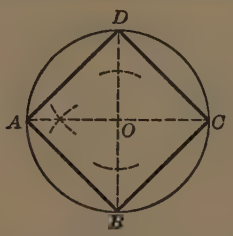
QUERY 1. In the figure of Theorem 1, how many degrees are there in the arcs AB, BC, CD, etc.? in $\measuredangle A$, B, C, etc.?

QUERY 2. An equilateral polygon of ten sides is inscribed in a circle. How many degrees are there in each angle?

QUERY 3. Is an equiangular polygon inscribed in a circle necessarily regular?

Construction 1

353. *Inscribe a square in a given circle.*



Given a circle whose center is O.

Required to inscribe a square in the circle.

Method. Draw the diameter BD, and construct the diameter $AC \perp$ to it. Draw the chords AB, BC, CD, and DA. Then $ABCD$ is the required square.

Proof

1. Arcs AB, BC, CD, and DA are equal.	1. § 181.
2. $\angle CDA = \angle ABC = \angle BCD = \angle DAB = 90°$.	2. § 229.
3. $\therefore ABCD$ is a square inscribed in $\odot O$.	3. §§ 123 and 200.

EXERCISES

1. Circumscribe a square about a given circle.

2. Inscribe a regular octagon in a given circle.

3. Inscribe a circle in a given square.

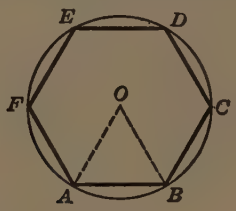
4. Describe a method by which regular polygons of 16 sides, 32 sides, 64 sides, etc. may be inscribed in a circle.

5. If the alternate vertices of a regular polygon of an even number of sides are joined, the polygon thus formed is regular. Prove.

6. Prove that the side of an inscribed square is $\frac{R}{2}\sqrt{2}$ from the center of the circle (R = radius).

Construction 2

354. *Inscribe a regular hexagon in a given circle.*



Given a circle whose center is O.

Required to inscribe a regular hexagon in the circle.

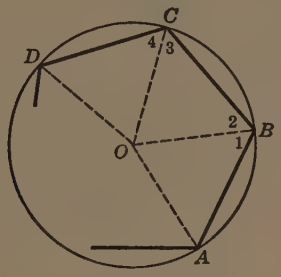
Method. Draw any radius OA. Then with A as the center and AO as the radius, describe an arc cutting the circle at B. With B as the center and OA as the radius, describe an arc cutting the circle at C. In like manner obtain points D, E, and F. Draw the chords AB, BC, CD, DE, EF, and FA. They will form the required regular hexagon.

Proof

1. Draw radii OA and OB.	1. § 51, Postulate 1.
2. $\triangle ABO$ is equilateral, and $\angle O = 60°$.	2. By construction, §§ 70 and 100.
3. $\therefore$ arc $AB = 60° = \frac{1}{6}$ of the circle.	3. § 225.
4. $\therefore$ arc $ABCDEF = \frac{5}{6}$ of the circle, and arc $FA = \frac{1}{6}$.	4. By construction and step 3.
5. Hence chord FA = chord AB.	5. § 185.
6. $\therefore$ polygon $ABCDEF$ is equilateral.	6. Steps 2 to 5.
7. $\therefore$ $ABCDEF$ is an inscribed regular hexagon.	7. § 350.

Theorem 2

355. *A circle can be circumscribed about any regular polygon.*



Given any regular polygon $ABCD \cdots$.

To prove that a circle can be described passing through the points A, B, C, D, ***etc.***

Method. Describe a circle through any three vertices. Then prove that the center of this circle is the same distance from a fourth vertex as it is from any of the three and hence the circle passing through three vertices passes through all the others.

Proof

1. Describe a circle through A, B, and C, point O being its center.	1. § 174.
2. Draw the radii OA, OB, OC, and the line OD.	2. § 51, Postulate 1.
3. $\angle 1 + \angle 2 = \angle 3 + \angle 4$.	3. § 350.
4. $\angle 2 = \angle 3$.	4. § 67.
5. $\therefore \angle 1 = \angle 4$.	5. § 50, Axiom 4.
6. $AB = CD$.	6. § 350.
7. $OB = OC$.	7. Radii of same $\odot$.
8. $\therefore \triangle AOB \cong \triangle COD$.	8. § 62.

9. $OA = OD$.	9. § 64.
10. $\therefore$ $\odot$ O passes through D. In like manner it can be proved that it passes through the other vertices.	10. Steps 1 to 9.

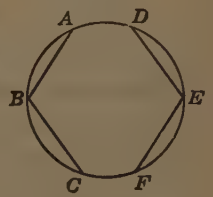
EXERCISES

1. Inscribe an equilateral triangle in a circle.

2. Inscribe a regular 12-sided polygon in a circle.

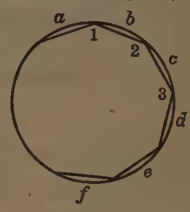
3. Circumscribe an equilateral triangle about a circle.

4. Circumscribe a regular hexagon about a circle.



5. If, in the figure, chords AB, BC, DE, and EF are equal, $\angle B = \angle E$. Prove.

6. An equiangular polygon inscribed in a circle is regular if it has an odd number of sides. Prove.



HINT. $\angle 1 = \frac{360 - a - b}{2}$. $\angle 2 = \frac{360 - b - c}{2}$.

$\therefore a = c$. Similarly, $a = c = e = g$, etc., (1)
and $b = d = f = i$, etc. (2)
If the number of sides is odd, one side at least of (1) will be found in (2), etc.

NEWTON AND THE RAINBOW

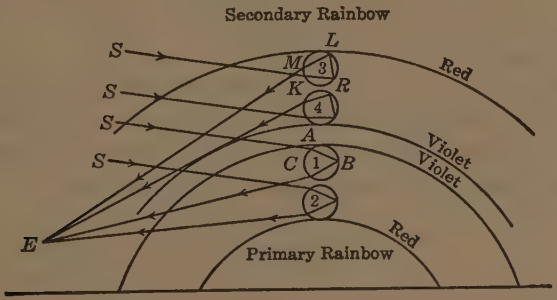
Sir Isaac Newton (1642–1727), besides being the greatest mathematician who ever lived, was also equally eminent as a physicist and an astronomer. No other discovery in science equals his discovery of the law of gravitation, named for him. He devoted much time to the study of the nature and properties of light. While so doing he examined the colors given out by thin films of water, and used to spend many hours on the porch of the house where he lived blowing soap bubbles and observing their changing colors. He spent so much time

at this that an observant old neighbor, who knew nothing of his real aims, confided to her friends that the poor man must be a lunatic.

Newton was the first to send a ray of sunlight (white light) through a prism and break it up into the seven primary colors — the colors of the rainbow. He then placed a second prism so that it undid the work of the first and reunited the colored light into a beam of white light again. This enabled him to explain satisfactorily the cause of the colors of the rainbow.

Everyone has seen a rainbow, but many have missed seeing the double rainbow — one higher than the other, but with the colors from red to violet reversed.

Before Newton the rainbow had been partially explained. Theodoric of Vriberg, about 1304–1311, had correctly traced the path of the light through the spherical drops of rain which formed the primary bow. Thus, the ray SA coming from the sun and entering drop 1 at A is bent — refracted. Within the drop, at B, it is reflected, and it is bent again on leaving the drop at C to go to the eye. The same is true of drop 2.



Marco Antonio de Dominis (1566–1624), archbishop of Spalato, improved on this explanation, and in addition traced the path of the light through the drops which formed the outer, or secondary, bow. Thus, the ray SK is bent (refracted) on entering the drop 3 at K, is reflected within the drop at R, reflected a second time at L, and bent again on leaving the drop at M to go to the eye at E. The same is true of drop 4.

Now, rainbows are seen in the morning or evening, when the sun is 40° or less above the horizon. The next step in the explanation was given by the mathematician Descartes in 1637. He used the law of refraction discovered by Snell of Holland and calculated the radius of the rainbow, finding that theory and observation agreed. But the colors in the two bows remained unexplained.

Newton's prism experiment enabled him to complete the explanation and to tell why the colors in the two rainbows are reversed. If the paths of the rays are followed, it will be seen that the direction of the bending of the light rays is opposite in the primary and the secondary bows. Hence the colors should come in opposite order in the two bows.

356. Center of polygon. The *center of a regular polygon* is the common center of its inscribed and circumscribed circles.

357. Radius of a regular polygon. The *radius of a regular polygon* is the radius of its circumscribing circle.

358. Apothem. The *apothem of a regular polygon* is the radius of its inscribed circle.

359. Angle at the center of polygon. The *angle at the center of a regular polygon* is the angle between the two radii drawn to the extremities of any side.

EXERCISES

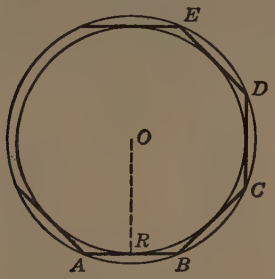
1. The apothem of an equilateral triangle is one half the radius. Prove.

2. The angle at the center of a regular polygon is the supplement of an angle of the polygon. Prove.

HINT. Let $ABCD \cdots$ be a regular polygon of n sides. How is $\angle D$ measured? $\angle C$?

Theorem 3

360. *A circle can be inscribed in any regular polygon.*



Given any regular polygon $ABCDE\cdots$.

To prove that a circle may be drawn tangent to AB, BC, CD, etc.

Method. Circumscribe a circle about the polygon, and prove that all the sides of the polygon are the same distance from the center. The theorem follows.

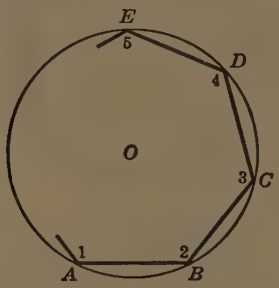
Proof

1. Circumscribe a circle about $ABCDE\cdots$. Let O be its center.	1. § 355.
2. Now $AB = BC = CD = DE$, etc.	2. By hypothesis.
3. These chords are the same distance from the center.	3. § 189.
4. Draw the apothem OR.	4. § 76.
5. With O as the center and OR as the radius, a circle can be drawn which is tangent to AB, BC, CD, etc.	5. § 211.

361. *Corollary. The centers of the inscribed and circumscribed circles of a regular polygon coincide.*

Theorem 4

362. *If a circle is divided into three or more equal arcs, the chords joining the adjacent points of division form a regular inscribed polygon.*



Given $\odot O$ **and points** A, B, C, D, E, **etc., dividing it into equal arcs. Given chords** AB, BC, **etc.**

To prove that $ABCDE\cdots$ ***is a regular inscribed polygon.***

Method. Prove that the inscribed polygon is equilateral and hence is regular.

Proof

1. Chord AB = chord BC = chord CD, etc.	1. § 185.
2. Hence the inscribed polygon $ABCDE\cdots$ is equilateral.	2. Step 1.
3. $\therefore$ the inscribed polygon $ABCDE\cdots$ is regular.	3. By hypothesis and § 352.

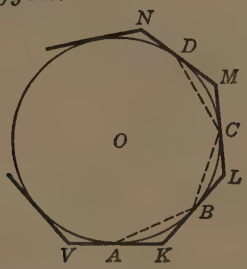
EXERCISES

1. Can an equilateral polygon of four sides be circumscribed about a circle? Is it regular? Prove.

2. Can an equiangular polygon of four sides be circumscribed about a circle? Is it regular? Prove.

Theorem 5

363. *If a circle is divided into three or more equal arcs, tangents at the points of division form a regular circumscribed polygon.*



Given $\odot O$ **and equal arcs** AB, BC, CD, **etc., with tangents at points** A, B, C, **etc. intersecting at** K, L, M, **etc.**

To prove that $KLM\cdots$ ***is a regular circumscribed polygon.***

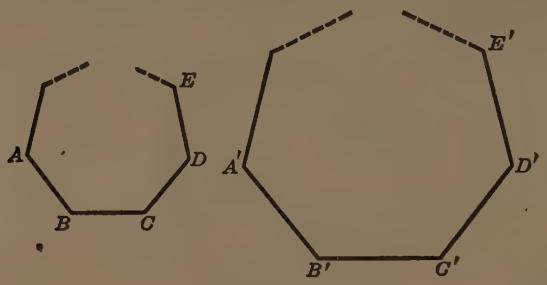
Method. Prove that the triangles formed by the chords and the adjacent tangents are congruent and isosceles. Then show that the angles of the circumscribed polygon are corresponding angles of these triangles and the sides are the sum of two equal sides of the triangles.

Proof

1. Draw chords AB and BC.	1. § 51, Postulate 1.
2. Chord AB = chord BC.	2. § 185.
3. $\angle BAK = \angle ABK = \angle CBL = \angle BCL$.	3. By hypothesis and § 235.
4. $\therefore \triangle ABK \cong \triangle BCL$ and each is isosceles.	4. §§ 66, 54.
5. $\therefore \angle K = \angle L$.	5. § 64.
6. In like manner $\angle L = \angle M$, etc.	6. Steps 1 to 5.
7. $VA = AK = KB = BL = LC$, etc.	7. § 64.
8. $\therefore VK = KL = LM$, etc.	8. § 50, Axiom 3.

Theorem 6

364. *If each of two regular polygons has n sides, the polygons are similar.*



Given the regular polygons $ABCDE\cdots$ and $A'B'C'D'E'\cdots$, each having n sides.

To prove that $ABCDE\cdots$ is similar to $A'B'C'D'E'\cdots$.

Method. Prove that each polygon is equilateral and hence all the ratios found between two sides, one from each polygon, are equal. Then express the value of an angle of each polygon in terms of n and show that these expressions are equal.

Proof

1. $AB = BC = CD$, etc., and $A'B' = B'C' = C'D'$, etc.	1. By hypothesis.
2. Hence $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}$, etc.	2. § 50, Axiom 6.
3. Each polygon is equiangular.	3. By hypothesis.
4. $\therefore \angle A = \frac{n-2}{n}$ st. $\angle$s, and $\angle A' = \frac{n-2}{n}$ st. $\angle$s.	4. § 158.
5. $\therefore \angle A = \angle A'$.	5. § 50, Axiom 1.

6. $\angle B = \angle B'$, $\angle C = \angle C'$, etc.	6. Steps 4 and 5.
7. $\therefore$ polygons $ABCDE\cdots$ and $A'B'C'D'E'\cdots$ are similar.	7. § 272.

EXERCISES ON THEOREMS 5 AND 6

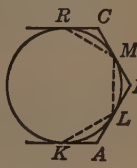
1. The side of the circumscribed equilateral triangle of a circle is twice that of the inscribed equilateral triangle. Prove.

2. Compare the exterior angle of a regular hexagon with its central angle.

3. An angle of a regular circumscribed polygon is the supplement of the angle measured by the smaller arc intercepted by two adjacent sides. Prove.

HINT. $\angle A$ is measured by $\frac{1}{2}[(360° - \text{arc } BC) - \text{arc } BC]$. How is $\angle ABC$ measured?

4. The diagonals of a regular pentagon form another regular pentagon. Prove.



HINT. Prove that the diagonals are equal and equally distant from the center.

5. An equiangular polygon circumscribed about a circle is regular. Prove.

HINT. Prove $\triangle AKL \cong \triangle BLM \cong \triangle CMR$, and that each is isosceles.

6. Do the diagonals from alternate vertices of a regular hexagon form another regular hexagon? Prove.

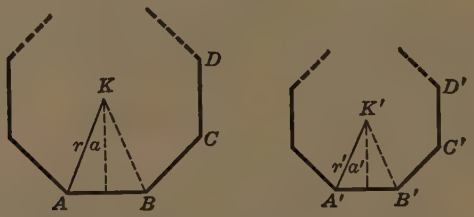
7. If an exterior angle of a regular polygon equals an interior angle, how many sides has the polygon? Prove your answer.

8. Find the number of degrees in each angle of a regular polygon of 20 sides.

9. Inscribe a circle in any assumed triangle which is neither equilateral nor isosceles.

Theorem 7

365. *The perimeters of two regular polygons of the same number of sides have the same ratio as their sides, their radii, or their apothems.*



Given the regular polygons $ABCD \cdots$ and $A'B'C'D' \cdots$, each of n sides, with perimeters p and p', radii r and r', and apothems a and a' respectively.

To prove that $\frac{p}{p'} = \frac{r}{r'} = \frac{a}{a'}$.

Method. In each polygon draw radii to the extremities of any one side, and prove that the triangles thus formed are similar. Then show that the two right triangles having the apothem as a side are similar. The theorem then follows from § 284.

Proof

1. Let K and K' be the common centers of the inscribed and circumscribed circles respectively. Draw KA, KB, $K'A'$, and $K'B'$.	1. § 356.
2. $\frac{r}{r'} = \frac{AB}{A'B'}$.	2. § 285.
3. $\therefore \triangle ABK \sim \triangle A'B'K'$.	3. § 282.
4. Also, $\frac{r}{r'} = \frac{a}{a'}$.	4. § 284.
5. Hence $\frac{r}{r'} = \frac{a}{a'} = \frac{AB}{A'B'}$.	5. § 50, Axiom 1.

6. $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}$. 6. § 272.

7. Hence $\frac{AB + BC + CD + \cdots}{A'B' + B'C' + C'D' + \ldots} = \frac{p}{p'} = \frac{AB}{A'B'}$. 7. § 263, 7.

8. $\therefore \frac{p}{p'} = \frac{r}{r'} = \frac{a}{a'}$. 8. Steps 5 and 7.

EXERCISES

1. The radius of a circle is 12″. Find the side, apothem, perimeter, and area of the circumscribed square.

2. The radius of a circle is 10″. Find the side, apothem, and area of the circumscribed equilateral triangle.

3. Find the side, apothem, and area of a regular hexagon circumscribed about a circle whose radius is 18″.

4. In a circle whose radius is 20″ find the side, apothem, perimeter, and area of a regular inscribed hexagon.

5. The radius of a circle is 72″. Find the side, apothem, perimeter, and area of the inscribed equilateral triangle.

6. $ABCDE$ is a regular inscribed pentagon. Draw the diagonals AD and AC, and find the number of degrees in each angle of $\triangle ADC$.

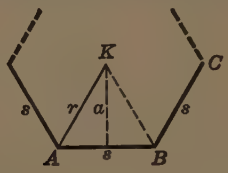
7. $ABCDEF$ is a regular hexagon. Draw the diagonals AE, AD, AC, and EC. Find the number of degrees in each angle of $\measuredangle AEF$, AED, and AEC.

8. $ABCDEFGH$ is a regular octagon. Draw AC, AD, AE, AF, and FC. Then find the number of degrees in each angle of $\measuredangle ABC$, ADC, ADE, and AFC.

9. The apothem and side of a regular polygon of nine sides are m and s respectively. Find the perimeter and the side of a regular polygon of nine sides whose apothem is c.

Theorem 8

366. *The area of a regular polygon is one half the product of its perimeter and its apothem.*



Given a regular polygon $ABC\cdots$ of n sides with apothem a and perimeter p.

To prove that the area of $ABC\cdots$ is $\frac{a \cdot p}{2}$.

Method. Draw radii to the extremities of each side and prove that the triangles thus formed are congruent. Then show that the area of one is $\frac{as}{2}$ and of all the triangles is $\frac{ap}{2}$.

Proof

1. Let K be the center of the inscribed circle and s one side of the polygon. Draw KA and KB.	1. § 360.
2. The polygon can be divided into n $\triangle$, each $\cong \triangle ABK$.	2. § 71.
3. The area of $\triangle ABK = \frac{a \cdot s}{2}$.	3. § 328.
4. Then the area of the n $\triangle$ $= n\frac{a \cdot s}{2}$, or $\frac{a}{2} n \cdot s$.	4. Steps 2 and 3.
5. But $n \cdot s = p$, the perimeter of the polygon.	5. Definition of perimeter.
6. $\therefore$ the area of $ABC\cdots$ $= \frac{a \cdot p}{2}$.	6. Steps 4 and 5.

367. *Corollary. The areas of two regular polygons of the same number of sides are to each other as the squares of their radii or the squares of their apothems.*

Proof

Let A and A' be the respective areas. Then, using the notation of § 366,

$$\frac{A}{A'} = \frac{ap}{a'p'} = \frac{a}{a'}\,\frac{p}{p'}.$$

But $\dfrac{a}{a'} = \dfrac{r}{r'} = \dfrac{p}{p'}.$ § 365

Hence $\dfrac{A}{A'} = \dfrac{a}{a'}\,\dfrac{a}{a'} = \dfrac{a^2}{(a')^2}.$ § 50, Axiom 7

Therefore $\dfrac{A}{A'} = \dfrac{a^2}{(a')^2} = \dfrac{r^2}{(r')^2}.$

EXERCISES

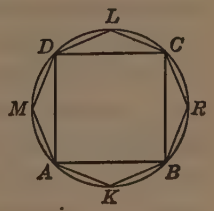
1. The radius of a circle inscribed in a square is 10. Find the area of the square.

2. The radius of a circle circumscribing a square is 20. Find the perimeter and area of the square.

3. The radius of a circle circumscribing a regular hexagon is 12. Find the perimeter and the area of the hexagon.

4. The radius of a circle inscribed in a regular hexagon is 18. Find the perimeter and the area of the hexagon.

5. Inscribe a square in a circle. Then inscribe a regular octagon.



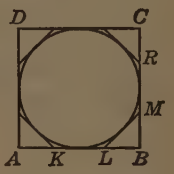
Prove that the perimeter of the octagon is greater than that of the square.

HINT. $AK + KB > AB$. Etc.

6. Inscribe a regular hexagon in a circle and then inscribe a regular 12-sided polygon in the same circle.

Prove that the perimeter of the regular 12-sided polygon is greater than the perimeter of the hexagon.

7. Circumscribe a square and a regular octagon about a circle and prove that the perimeter of the octagon is less than the perimeter of the square.



HINT. $LB + BM > LM$. Etc.

8. Circumscribe a regular hexagon and a regular 12-sided polygon about a circle and prove that the perimeter of the hexagon is greater than the perimeter of the 12-sided polygon.

368. Measurement of the circle. The measurement of the circle involves two problems: one is finding the length of the circle; the other is finding the area.

The length of a circle is called its *circumference*.

The word "length" has up to this point been used only in connection with straight lines. Now, a straight line of unit length cannot be applied to a circle to measure its length directly; the measurement must be carried out indirectly.

It is easily seen from Exercises 5 to 8, above, that the perimeter of an inscribed regular polygon is less than the circumference and that the perimeter of the regular circumscribed polygon is greater than the circumference.

Now, a regular polygon can be inscribed in a circle and its perimeter measured. So also can a regular polygon be circumscribed about a circle and its perimeter measured. Further, if a regular polygon of twice as many sides be inscribed, its side and its perimeter can be computed in terms of the radius. A convenient formula for this will

be derived later. Similarly, if a regular polygon of twice the number of sides be circumscribed, its side and its perimeter can be computed in terms of the radius.

If this process be continued and the results compared, it will be found that, as the number of sides is successively doubled, the perimeters of the successive regular inscribed polygons increase and the perimeters of the regular circumscribed polygons decrease. As this goes on, the difference between the perimeters of the corresponding inscribed and circumscribed polygons becomes less and less. If the computation be carried far enough, the difference can be made as small as we please. This means that the perimeters of the inscribed and circumscribed polygons approach the same value, and that value is the length of the circle, or the circumference of the circle. As the inscribed polygons approach the circumference their apothems increase and approach the radius.

Moreover, the areas of the successive inscribed polygons can be computed, and they will be found to increase as the number of sides increases. Similarly, the areas of the regular circumscribed polygons will be found to decrease as the number of sides increases. The difference in area of the corresponding inscribed and circumscribed polygons can be made as small as we please. This means that the areas of the inscribed and circumscribed polygons, as the number of sides increases, approach the same value, and that value is the area of the circle.

The preceding explanation does not prove, but brings out the meaning of, the following theorem, which is used as a basis for the computation of the value of the ratio of the length of a circle to the length of its diameter.

Theorem 9

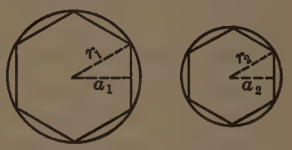
369. *If a series of regular polygons be inscribed in a circle, each, after the first, having twice as many sides as the preceding one, the perimeters of the polygons increase as the number of sides increases and approach as nearly as we please to the circumference. At the same time the apothem approaches the radius of the circle.*

This theorem, the truth of which is easily grasped, will be assumed without proof. A demonstration of it will not be given, as it is usually omitted in a first course in geometry.

370. *Corollary 1. The circumferences of two circles are in the same ratio as their radii or their diameters.*

Proof

The perimeters, p_1 and p_2, of two polygons of the same number of sides inscribed in two different circles whose circumferences are C_1 and C_2 are in the same ratio as their apothems, a_1 and a_2. That is,



$$\frac{a_1}{a_2} = \frac{p_1}{p_2}. \qquad \S\ 365$$

If the number of sides of the polygons be repeatedly doubled, p_1 approaches as nearly as we please to C_1, p_2 to C_2, a_1 to r_1, and a_2 to r_2. (§ 369.)

Hence $$\frac{a_1}{a_2} = \frac{p_1}{p_2} = \frac{r_1}{r_2} = \frac{C_1}{C_2};$$

also $$\frac{C_1}{C_2} = \frac{2\,r_1}{2\,r_2} = \frac{D_1}{D_2}.$$

This corollary proves that circles of different sizes follow the same rules as do similar polygons. We may therefore say that, as in similar polygons, the corresponding parts of two circles have a constant ratio to each other.

371. *Corollary 2. The ratio* $\frac{C}{D}$, *any circumference divided by its diameter, is the same for all circles.*

Proof

$$\frac{C_1}{C_2} = \frac{D_1}{D_2}, \qquad \text{§ 370}$$

Hence $$\frac{C_1}{D_1} = \frac{C_2}{D_2}. \qquad \text{§ 263, 3}$$

That is, the number obtained when any circumference is divided by its diameter equals the number obtained when any other circumference is divided by its diameter. This number is represented by the Greek letter π (pronounced pī), that is,

$$\frac{C}{D} = \pi.$$

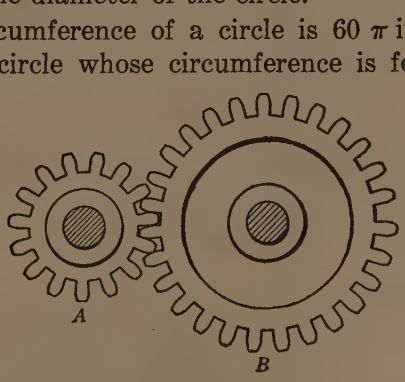
$$\therefore C = \pi D \quad \text{or} \quad 2\pi R.$$

EXERCISES

1. The radius of a circle is 5 ft. Show that the circumference is 10π ft.

2. The circumference of a circle is 24π in. Find the radius and the diameter of the circle.

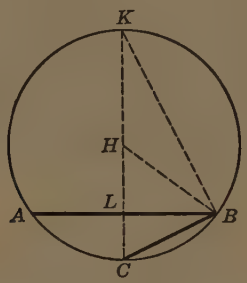
3. The circumference of a circle is 60π in. Find the radius of a circle whose circumference is four times as great.



4. How many times does cogwheel A turn while B turns once? As A turns 30 times, how many turns does B make?

Problem 1 (*Computation*)

372. *If s is the side of any regular inscribed polygon, x the side of a regular inscribed polygon having twice as many sides as the first, and R the radius of the circle circumscribing both, express x in terms of s and R.*



Given $\odot H$; AB, or s, and BC, or x, the sides of regular inscribed polygons of n sides and $2n$ sides respectively; and the radius R.

Required to express x in terms of s and the radius R.

Solution. Draw the diameter $CK \perp$ to AB and cutting it at L. Draw BK and BH.

Then CBK is a rt. $\angle$. § 230

Then $x^2 = \overline{CB}^2 = CK \cdot CL$, (1) § 288 (*b*)

and $x = \sqrt{2R \cdot CL}$. (2)

Now $CL = R - LH$, (3)

and, since $LB = \frac{s}{2}$, $LH = \sqrt{R^2 - \frac{s^2}{4}}$. (4) § 290

From (3) and (4), $CL = R - \sqrt{R^2 - \frac{s^2}{4}}$. (5)

From (2) and (5), $x = \sqrt{2R^2 - 2R\sqrt{R^2 - \frac{s^2}{4}}}$. (6)

If $R = 1$, (6) becomes $x = \sqrt{2 - 2\sqrt{1 - \frac{s^2}{4}}}$.

EXERCISES

1. If, in the formula of Problem 1, $R = 1$ and $s = \sqrt{3}$, show that $x = 1$.

2. Show that the side of a regular inscribed dodecagon is expressed by the formula

$$R\sqrt{2 - \sqrt{3}} = (.5176+)R.$$

HINT. Here $s = R$.

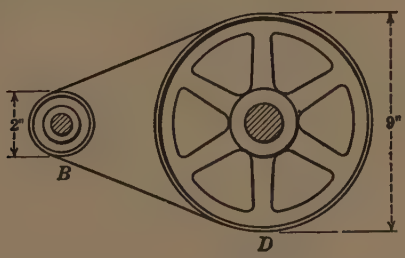
3. Show that the perimeter of a dodecagon in a circle whose radius is 1 is 6.2112+.

4. Show that the result of Exercise 3 gives an approximate value for π of 3.1056+.

5. The side of an inscribed square is $R\sqrt{2}$. Use the formula and show that the side of a regular inscribed octagon is expressed by the formula

$$R\sqrt{2 - \sqrt{2}} = (.765+)R.$$

6. A circle is inscribed in a square whose side is 2 ft. 4 in. Find the circumference in terms of π.



7. A continuous belt runs around two pulleys, B and D, which are mounted so as to turn freely and so that if either wheel rotates the other does also. If pulley D makes 1200 R. P. M. (revolutions per minute), how many does B make?

373. Computation of π. Problem 1 can be applied to determine the value of π to any desired degree of accuracy. In carrying out the computation it is simpler to use a circle whose radius is 1 and start with a regular inscribed hexagon. Then, as in Exercises 2 and 3 on page 377, one application of Problem 1 gives the side and the perimeter of a regular polygon of 12 sides. Dividing the perimeter thus obtained by the diameter, 2, gives an approximate value of π. Using the value .5176 thus obtained for the side of a regular dodecagon, a second application of Problem 1 gives the side of a regular inscribed polygon of 24 sides. Its perimeter gives values which are more nearly correct.

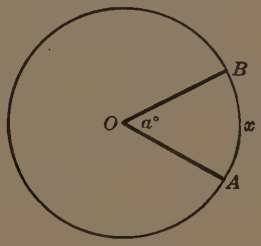
Repeated application of Problem 1 gives the results in the following table, in which it is more important to understand the method of obtaining the increasingly more accurate values of π than to make the computation.

n	R	s	P	$P \div D$, or π
6	1	1	6	$6 \div 2 = 3$
12	1	.51763809	6.21165708	3.10589+
24	1	.26105238	6.26525722	3.13262+
48	1	.13080626	6.27870041	3.13935+
96	1	.06543817	6.28206396	3.14103+
192	1	.03272346	6.28290510	3.14145+
384	1	.01636228	6.28311544	3.14155+
768	1	.00818126	6.28316941	3.14158+

In the table above, n denotes the number of sides of a regular inscribed polygon, R the radius of its circumscribed circle, s the length of one side, P the perimeter, and D the diameter. $P \div D$ will be an approximate value of π.

Theorem 10

374. *The length of an arc of a circle in linear units is to the circumference as the number of degrees in the arc is to 360.*



Given ⊙ O, whose radius is R, and the arc AB, whose length is x and which contains $a°$.

To prove that $x : 2\pi R = a : 360$.

Method. Express the circumference in terms of π and R. Then compare the number of degrees in the angle about point O and its arc, the circumference, with the number of degrees in $\angle a$ and its arc, x.

Proof

1. The circumference is $2\pi R$.	1. § 371.
2. The circumference has a central $\angle$ of 360°.	2. § 44, Example 2.
3. The arc x has a central $\angle$ of $a°$.	3. By hypothesis.
4. $\therefore x : 2\pi R = a : 360$.	4. § 225.

QUERY 1. Why are linear units specified in Theorem 10 for the measurement of the circumference and the arc x?

QUERY 2. In two unequal circles, are equal central angles represented by an equal number of degrees of the intercepted arcs or by an equal linear length of the intercepted arcs? What difference does it make which measure is used?

EXERCISES

Unless otherwise directed, the value 3.1416 is to be used for π in all computations on the circle.

1. Find the length of an arc of 24° in a circle whose radius is 4.

2. An arc of a circle is 8π in. long and contains 20°. Find the radius of the circle.

3. The radius of a circle is 20 and the length of an arc is 5π. How many degrees are there in the arc?

4. A wheel stands in water which reaches halfway to the center of the hub. How many degrees are there in the arc bounding the submerged portion?

5. How long does it take the minute hand of a clock to move through 135°?

6. How long does the hour hand of a clock require to move through 84°?

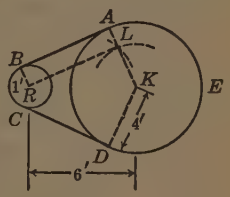
7. The hands of a clock are 56 in. and 63 in. long respectively. How far does the outer extremity of the minute hand travel in the time from 8.30 A.M. to 2.20 P.M.? the outer extremity of the hour hand? (Use $\frac{22}{7}$ for π.)

8. A bicycle wheel is 28 in. in diameter. How many revolutions does it make per hour when the bicycle is ridden 20 mi. per hour? (Use $\pi = \frac{22}{7}$.)

9. The radius of a wheel on a truck is 21 in. If the truck runs 35 mi. per hour, how many revolutions does the wheel make per minute? (Use $\frac{22}{7}$ for π.)

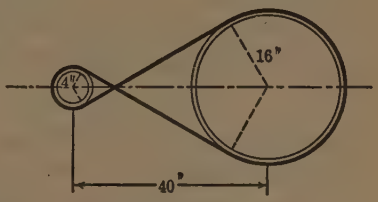
10. The tires on an automobile are 32 in. How many times do they revolve per second when the car is driven 55 mi. per hour?

11. The centers of two pulleys (wheels) are 6 ft. apart. The radii are respectively 1 ft. and 4 ft. What is the length of the belt which runs around the two?



HINT. In the adjacent figure the length of the belt is AB plus arc BC plus CD plus arc DEA. Draw KA, KR, and RB. Then draw RL parallel to AB, cutting KA at L.

12. Through how many degrees does a point on the equator revolve about the axis of the earth in 12 min.? Through how many miles? (Use 3960 mi. as the radius of the earth.)

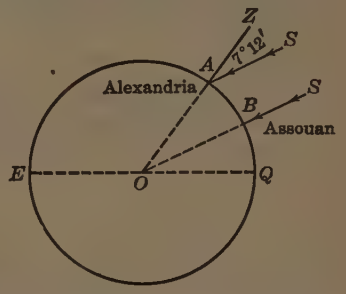


13. The radii of two pulleys are 16 in. and 4 in. respectively and their line of centers is 40 in. Find the length of a belt passing around them and crossed as in the adjacent figure.

THE EARTH'S CIRCUMFERENCE WAS MEASURED TWENTY-TWO HUNDRED YEARS AGO

The fact that the earth's shadow on the moon at the time of a lunar eclipse is round was one thing which led the ancients to the belief that the earth was a sphere. They next wished to know its circumference. The first successful attempt to measure the earth's circumference was made about 250 B.C. by Eratosthenes, the librarian of the University of Alexandria, and a friend of the great Archimedes. Eratosthenes used the fact expressed in Theorem 10. He observed that on the longest day in summer — what we now call June 21 — the sun shone directly down in a well in Syene (Assouan), a city located at the first cataract of the Nile. On the same day at Alexandria, nearly north of Syene, the sun did not appear overhead but slightly to the south in such a position that

$\angle ZAS$ was 7° 12′. The sun is so far away that the lines AS and BS are practically parallel. Since $\angle ZAS = \angle AOB$, all that remained was to measure the length of arc AB. This was about 5000 stadia, the unit of length used. One stadium was equal to 517 ft. Hence (by Theorem 10) $5000 : C = 7° 12′ : 360°$. This gives 250,000 stadia or about 24,500 mi. for the circumference of the earth, which is an excellent result for those days. Hence over seventeen centuries before Columbus the world was known to be a sphere and its circumference had been accurately measured.



375. Area of a circle. The area of a parallelogram which is not a rectangle, of a triangle, or of a trapezoid depends in a simple manner on the area of a rectangle, though no one of the three figures can be completely divided into rectangles. The area of a circle is very much more difficult to measure than that of any figure bounded by straight lines. The part of a plane bounded by a circle cannot be divided into squares or rectangles, however small they may be, nor can its area be made to depend in any simple way on either.

The area of a circle can, however, be made to depend on that of an isosceles triangle. Let AB be the side of a regular inscribed polygon of n sides and OK the apothem. The area of the polygon will be the area of n triangles like AOB. This has been shown to be one half the perimeter of the polygon multiplied by the apothem (Theorem 8). If the number of sides of the polygon be repeatedly doubled, the area of the polygon increases and is

made up of n such isosceles triangles. Moreover in this process of doubling the sides the perimeters approach the circumference and the apothems approach the radius (Theorem 9), and the areas approach the area of the circle. Therefore the expression for the area of a polygon, $A = \frac{Pa}{2}$, becomes more and more nearly $A = \frac{CR}{2}$. The proof here outlined can be made formal and exact. This is not usually done in a first course in geometry. Therefore Theorem 11 will be assumed.

EXERCISES

1. A locomotive runs 45 mi. per hour. How many revolutions per second do the 6-foot drive wheels make?

2. If a circle is divided into n equal arcs, is a sector bounded by one arc and two radii including a small angle approximately a triangle?

3. What is the altitude, approximately, of the triangle of Exercise 2? the base? the area?

4. What, then, is the approximate area of the entire circle of Exercise 2?

5. If n, the number of equal arcs, is doubled, are the answers to the preceding exercises changed?

6. Suppose the number of arcs is indefinitely great, what bearing have these conclusions on the truth of Theorem 11, which follows?

7. Is the method of reasoning followed in Exercises 2 to 6 similar to the reasoning used in finding the value of π? If this reasoning led to a correct result in the case of π, is it reasonable to suppose that it will lead to a correct formula for the area of a circle?

Theorem 11

376. *The area of a circle equals one half the product of the circumference and the radius.*

377. ***Corollary 1.*** *The area of a circle is* πR^2.

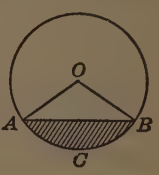
Proof

1. $C = \pi D = 2\pi R$.	1. § 371.
2. $A = \frac{CR}{2}$.	2. § 376.
3. $\therefore A = \frac{2\pi R \cdot R}{2} = \pi R^2$.	3. § 50, Axiom 7.

378. ***Corollary 2.*** *The areas of two circles are to each other as the squares of their radii.*

379. ***Corollary 3.*** *The area of a sector of a circle equals one half the product of its radius and its arc.*

380. ***Corollary 4.*** *The area of a segment whose arc is less than a semicircle equals the area of the corresponding sector minus the area bounded by two radii and the chord of the segment.*



From the adjacent figure, segment ABC equals sector $OACB - \triangle OAB$.

QUERY 1. How is the area of a segment whose arc is greater than a semicircle found?

QUERY 2. What theorem about similar polygons corresponds to Theorem 11?

EXERCISES ON CIRCLES

1. The radius of circle O is 14′, and arc ACB equals 90°. Find the area of segment ACB and of segment ABK. (Use $\pi = \frac{22}{7}$.)

2. If the radius of a circle is 12″, find the circumference and the area.

3. The area of a circle is 9π. Find the diameter.

4. The circumference of a circle is 8π. Find the diameter and the area.

5. The radii of two circles are respectively 9″ and 12″. Find the diameter of a circle equivalent to their sum.

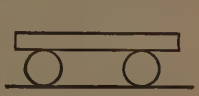
6. The radius of a circle is 12′. Find the area of a sector of the circle whose arc is 50°.

7. Find the area of a sector of 24° in a circle whose radius is 20″.

8. Find the radius of a circle whose circumference equals the sum of the circumferences of the two circles whose radii are 5″ and 12″ respectively.

9. The radii of two circles are respectively 3′ and 18′. Show, without finding the area of either, that the area of the larger is 36 times the area of the smaller.

10. A slab of stone is moved by means of two 5-inch rollers placed as shown. How far has the slab moved when the rollers have revolved once?



11. The radius of a circle is 7′. An arc in it is 6′ long. Find the area of the corresponding sector. (Use $\frac{22}{7}$ for π.)

12. The hour hand of a clock is 6′ long. Find the area of the sector over which it moves in the time from 12.40 P.M. to 3.15 P.M.

13. The radius of a circle is 10′. Find the radius of a concentric circle dividing the portion of the plane bounded by the first into two equal parts.

14. The area of a circle is divided by two concentric circles into three equivalent parts. The radius of the smallest circle is 8. Find the radii of the other circles.

15. Semicircles are described on the sides of a right triangle as diameters. Prove that the area of the semicircle on the hypotenuse equals the sum of the areas of the other two semicircles.

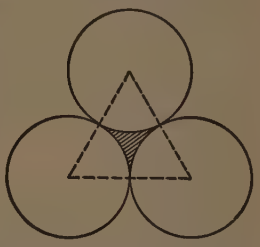
16. Semicircles are described on the sides of a right triangle as in the adjacent figure. Prove that the sum of the two shaded areas equals the area of the triangle (Theorem of Hippocrates, about 460 B. C.)

17. Find the area of a segment whose arc is 90°, the radius of the circle being 10′.

18. Find the area of a segment whose arc is 30° in a circle of radius 12.

19. Let the radius of a circle be 20″ and x be the number of degrees in the arc of a segment. Find the area of the segment when x is 45°; 150°; 300°.

20. Three equal circles, of radius 10″, are tangent each to the other two. Find the area of the triangular portion of the plane bounded by arcs of the three circles.



21. Show that the area of the segment bounded by a side of an inscribed square and its arc is $\frac{R^2}{4}(\pi - 2)$.

22. Show that the area of the segment of any circle bounded by a side of a regular inscribed hexagon and its subtended arc is

$$R^2\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right).$$

381. The area of a segment by trigonometry. The area of a segment for any angle can be found accurately by the use of trigonometry.

EXAMPLE

Find the area of a segment whose central angle is 72° in a circle whose radius is 12.

Solution. The area of sector AOB is

$$\pi \cdot 144 \cdot \tfrac{72}{360} = 28.8\,\pi = 90.478.$$

The area of $\triangle AOB = \frac{1}{2}\,12 \cdot 12 \cdot \sin 72°$

$$= 72(.9511) = 68.4792.$$

(Exercise 1, § 314)

$$90.478 - 68.4792 = 21.999.$$

382. Approximate formula for the area of a segment of a circle. Frequently in practice it is much easier to measure AB, the chord, or the base of the segment in the adjacent figure, and RK, the mid-perpendicular of chord AB, than to measure the arc AKB or its central angle. Let b denote AB and h denote RK. Then the area of $AKBR$ is

$$\frac{h^3}{2b} + \frac{2bh}{3}.$$

This formula is approximate, but gives very accurate results.

EXERCISES

1. Test the accuracy of the formula of § 382 thus: Find the area of a semicircle in the usual manner, then by the use of the formula, and compare the two results.

2. The radius of a circle is 10′. The height of a segment in it is 2′. Find the area of the segment.

3. Test the accuracy of the formula of § 382 on a segment whose arc is 90°.

4. Find by § 380 and by § 382 the area of the segment whose chord is the side of a regular inscribed hexagon, and compare the two results.

5. Find by trigonometry the area of a segment whose central angle is 24° in a circle whose radius is 40.

6. The radius of a circle is 20. Find the area of a segment subtended by the side of a regular decagon.

7. Find the area of a sector of a circle whose central angle is 60° in a circle whose radius is 12 in.

8. An arc of a circle whose radius is 40 in. contains 60°. Find the area bounded by the arc and its chord.

9. An arc of a circle whose radius is 32 in. contains 150°. Find the area bounded by the arc and its chord.

10. The radius of a circle is $5\sqrt{2}$ in. Find the radius of a circle having half the area of the first.

11. A circle whose radius is 8 ft. is divided into two equal parts by a concentric circle. Find the radius of the smaller circle.

12. A barn 60 ft. by 80 ft. stands in a meadow. A horse is tied to one corner by a rope 100 ft. long. Find the area over which he can graze.

383. Mean and extreme ratio. A line is divided by a point in mean and extreme ratio when one segment is a mean proportional between the whole line and the other segment.

$AB : AR = AR : RB$

Fig. 1

$AB : BK = BK : AK$

Fig. 2

There are two cases: (*a*) when the point R is on the given line AB (Fig. 1); (*b*) when the point K is on the given line AB produced (Fig. 2).

As case (*b*) is not used in elementary geometry, it will be given no further consideration here.

Problem 2 (*Computation*)

384. *Determine the numeric values of the two segments of a line of length m divided internally in mean and extreme ratio by a point on the line.*

Given the line *AB* of length *m*, and the point *K* dividing the line in mean and extreme ratio.

Required to express the values of the segments AK and KB in terms of m.

Solution. Let K be the point which divides AB internally in mean and extreme ratio. Then, by the definition of § 383,

$$AB : AK = AK : KB. \tag{1}$$

Let $AK = x.$

Then $KB = m - x.$

Substituting in (1), $m : x = x : (m - x).$ (2)

Then $x^2 = m^2 - mx,$ (3)

or $x^2 + mx = m^2.$ (4)

Completing the square, $x^2 + mx + \frac{m^2}{4} = m^2 + \frac{m^2}{4} = \frac{5\,m^2}{4}.$ (5)

$$x + \frac{m}{2} = \pm \frac{m}{2}\sqrt{5}. \tag{6}$$

$$\therefore x = -\frac{m}{2} + \frac{m}{2}\sqrt{5}, \quad \text{or} \quad \frac{m}{2}(-1 + \sqrt{5}), \tag{7}$$

and $$x = -\frac{m}{2} - \frac{m}{2}\sqrt{5}, \quad \text{or} \quad \frac{m}{2}(-1 - \sqrt{5}). \tag{8}$$

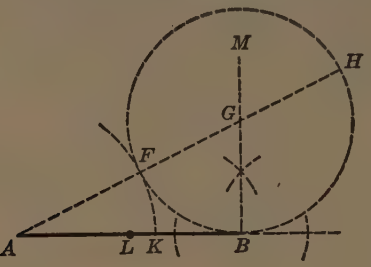
Since (8) gives a negative value of x, it is neither AK nor KB of the figure. Therefore (7) gives the length of AK.

Hence $KB = m - x = m - \left(-\frac{m}{2} + \frac{m}{2}\sqrt{5}\right) = \frac{3\,m}{2} - \frac{m}{2}\sqrt{5},$ (9)

or $$KB = \frac{m}{2}(3 - \sqrt{5}).$$

Construction 3

385. *Divide a given line internally in extreme and mean ratio.*



Given the line AB.

Required to locate a point K ***on the line*** AB ***such that*** $AB : AK = AK : KB$.

Construction. Bisect AB at L. Construct $BM \perp$ to AB at B. Lay off BG equal to BL. Construct a circle with G as the center and BG as the radius. Draw AG, cutting the circle in F and H. Lay off AK on AB equal to AF. Then K is the required point.

Proof

1. $AH : AB = AB : AF$.	1. § 294.
2. Then $(AH - AB) : AB = (AB - AF) : AF$.	2. § 263, 6.
3. Now $AB = FH$.	3. Why?
4. Also, $AF = AK$.	4. Why?
5. Then $(AH - FH) : AB = (AB - AK) : AK$.	5. Substituting from steps 3 and 4 in step 2.
6. But $AH - FH = AF$, or AK.	6. By construction.
7. Also, $AB - AK = KB$.	7. By construction.
8. Then $AK : AB = KB : AK$.	8. Substituting from steps 6 and 7 in step 5.
9. Hence $AB : AK = AK : KB$.	9. Why?
10. $\therefore$ K is the required point.	10. § 383.

THE GOLDEN SECTION

The construction of § 385, called the "golden section" by the Greeks, was of great interest to the early Greek geometers. Plato (429–347 B.C.) was the first to discuss and solve the problem. Later Eudoxus (408–355 B.C.), who is usually ranked as third among the Greek mathematicians, added several other theorems concerning it. Euclid, in the thirteenth book of his "Elements," has five theorems on the "golden section." This name was used by the Greeks to designate what now is usually called mean and extreme ratio. The Greeks knew little of modern algebra and nothing of the modern, or Hindu, number system. Hence their proofs are very much more difficult in appearance than the proof given of Problem 2. The actual geometric construction, however, follows that of § 385.

The division of a line as in § 385 has received other names than the "golden section." It has been claimed in modern times that this ratio is found in the structure of plants, in crystals, and in the human body. Its presence in these cases has been held to be one of the bases for the beauty of these forms. Psychological tests establish only one fact in connection with these claims, and that is for a rectangle. These tests indicate that a rectangle whose length is b and whose width is a is more pleasing to the eye if $(a+b):b=b:a$. This fact is often used in art work.

EXERCISES

1. A belt runs around two pulleys whose radii are 18 in. and 8 in. respectively. If the larger pulley revolves 800 times per minute, how many revolutions per second does the smaller make?

2. The tips of the teeth of a circular saw 4 ft. in diameter run 12,000 ft. per minute. How many revolutions does the saw make per second?

3. The hands of a tower clock are 5 ft. and 7 ft. long respectively. How far does the outer extremity of each hand move in the time from 8.30 A.M. to 2.15 P.M.?

Construction 4

386. *Inscribe a regular decagon in a given circle.*

Fig. 1

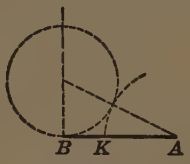


Fig. 2

Given a circle whose center is *A*.

Required to inscribe a regular decagon in the circle.

Method. Draw any radius *AB*. Then divide *AB* internally in mean and extreme ratio as indicated in Fig. 2. With *AK*, the larger segment, as the radius and *B* of Fig. 1 as the center, describe an arc cutting the circle at *C*. With *C* as the center and *AK* as the radius, describe an arc cutting the circle at *F*. In like manner obtain points *G*, *H*, *L*, *M*, *O*, *P*, and *Q*. Draw chords *BC*, *CF*, etc. These chords form the required decagon.

Proof

1. In Fig. 1 draw AB and lay off AK on radius AB equal to AK of Fig. 2 and draw KC.	1. § 51, Postulate 1, and § 385.
2. In Fig. 1, $AB : AK = AK : KB$.	2. By construction.
3. But $BC = AK$.	3. By construction.
4. Hence $AB : BC = BC : KB$.	4. § 50, Axiom 7.
5. Now $\angle B = \angle B$.	5. Identical.
6. Hence $\triangle ABC \sim \triangle CBK$.	6. § 281.
7. Then $AB : AC = CB : CK$.	7. § 285.
8. But $AB = AC$.	8. Radii of same $\odot$.

9. Hence $CB = CK$.	9. Steps 7 and 8.
10. Then $CK = AK$.	10. Steps 3 and 9.
11. Now $\angle BAC + \angle ABC + \angle BCA = 180°$.	11. § 100.
12. $\angle ABC = \angle BCA$.	12. Step 8 and § 67.
13. $\angle KCA = \angle A$.	13. Step 10 and § 67.
14. $\angle BCK = \angle A$.	14. Step 6 and § 272.
15. $\angle BCA = \angle KCA + \angle BCK$.	15. § 50, Axiom 2.
16. $\angle BCA = 2 \angle A$.	16. Steps 13, 14, and 15.
17. $\angle ABC = 2 \angle A$.	17. Steps 12 and 16.
18. $\angle A + 2 \angle A + 2 \angle A = 180°$.	18. Steps 11, 16, and 17.
19. $\angle A = 36°$.	19. Step 18.
20. $\therefore$ arc BC is one tenth of the circle.	20. § 225.
21. $\therefore$ $BCFGH \cdots$ is a regular inscribed decagon.	21. Steps 1 to 20 and § 352.

387. *Corollary 1. If alternate vertices of a regular inscribed decagon be joined, the polygon formed is a regular pentagon.*

388. *Corollary 2. If BC is the side of a regular hexagon and BK the side of a regular decagon, then KC is the side of a regular 15-sided polygon.*

HINT. In the adjacent figure prove $\angle KOC = 24°$.

EXERCISES

1. How can a regular polygon of 30 sides be constructed? of 60 sides? of 120 sides?

2. How many regular polygons of less than 100 sides can be constructed by the use of §§ 372, 386, and 387?

3. Write in order, the smallest first, the series of numbers involved in the answer to the preceding exercise.

THE NUMBER OF INSCRIBED REGULAR POLYGONS

It has been shown how to construct regular inscribed polygons of 3, 5, 6, 10, and 15 sides. It has also been shown that by bisecting arcs regular polygons having 2, 4, 8, $\cdots$ times as many sides as the preceding can be inscribed. These facts can be compactly stated as follows: a circle can be divided into n equal parts if $n = 2^m$, 3, 5, or the product of any two or three of these numbers. Here 2^m takes care of the doubling of the number of sides by the bisection of arcs. All this was known to the later Greek geometers. Whether there were other regular inscribed polygons than these was unknown until Gauss (1777–1855), when only nineteen years old, showed that a circle could be divided into 17 equal parts. Later he enlarged this series of numbers by showing that the division of the circle into n equal parts is *possible for* every prime number $n = 2^{2^m} + 1$ *but impossible for* all other prime numbers and their powers. Assigning values to m, we obtain corresponding values of n as follows:

$m = 0$	1	2	3	4
$n = 3$	5	17	257	65537

in which all the values of n are prime. Hence inscribed regular polygons with these five values of n are possible. This means that regular polygons of 7, 11, 13, 19, 23, $\cdots$ equal parts cannot be inscribed in a circle. From 3 to 257 inclusive there are 56 prime numbers and it is entirely unexpected to find that for only the four of these which are given in the table is a regular inscribed polygon possible.

To find other values of n and prove them prime is a problem in the theory of numbers. Very recently it has been shown that if $m \gtrdot 100$ there are 24 values of n, that if $m \leqq 1000$ there are 52 values of n, and that if $m \gtrdot 100{,}000$ there are 206 values of n. Thus the five values of n in the table have been extended to 206. Undoubtedly for values of m greater than 1000 the formula of Gauss gives other prime numbers. How many there are beyond the 206 already known to exist is, however, an unsolved problem.

MISCELLANEOUS EXERCISES

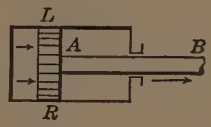
1. A central angle whose intercepted arc is equal to the radius of the circle is called a radian. Show that one radian equals 57.29+ degrees.

2. Show that π radians equal 180°.

3. How many degrees are there in $\frac{\pi}{2}$ radians? $\frac{\pi}{6}$? $\frac{3\pi}{2}$? $\frac{2\pi}{3}$? $\frac{\pi}{3}$?

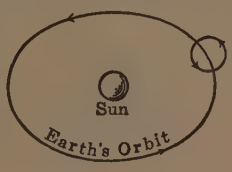
4. How many radians are there in 90°? 135°? 60°? 240°? 30°? 45°? (Give answers in terms of π.)

5. The diameter of a cylinder is 24″. Find the pressure on the piston LR when the steam pressure is 90 lb. to the square inch. What is the thrust along the piston rod AB?



6. The earth turns on its axis once in about 24 hr. Through how many degrees does a point on the equator rotate in 7 hr.? Through how many miles does it rotate about the earth's axis? ($R = 3960$ mi.)

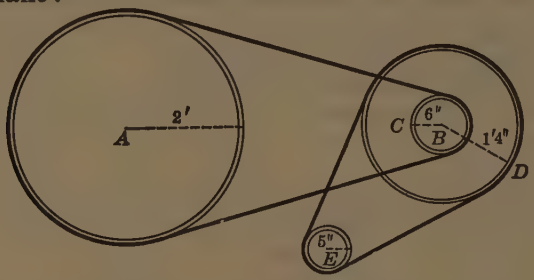
7. The adjacent figure gives a view of the earth's path around the sun as seen from the north. The arrows show the direction of the earth's motion around the sun and the direction of the earth's rotation on its axis. Show that in a year of 365 days the earth rotates 366 times on its axis.



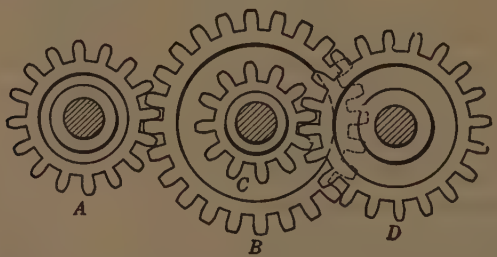
8. From the result of Exercise 7 show that the earth rotates on its axis once in about 23 hr. 56 min., not 24 hr.

9. If the earth rotates on its axis once in 23 hr. 56 min., how many miles does a point 30° north of the equator move in 10 hr. 20 min.? ($R = 3960$ mi.)

10. In the figure below, point D on the belt travels 16 ft. per second. How many revolutions per minute does each wheel make?



11. In the figure below, the cogwheels B and C turn together, being made of a single piece of iron. While A revolves 24 times how many times does the wheel B revolve? the wheel C? the wheel D?



12. While D turns 18 times how often does C turn? B? A?

13. While B revolves 36 times, how often does A revolve? C? D?

EXERCISES ON MENSURATION OF CIRCLES

1. The radius of a circle is 4 ft. 6 in. Find its circumference.

2. The diameter of the earth is 7917.6 mi. Find its circumference.

3. A wagon wheel is 3 ft. 4 in. in diameter. How far does it travel while revolving 2400 times?

4. A wheel turns 87 times in traveling 564 yd. Find its diameter.

5. A balloon tire on an automobile is 32″ in diameter. How many times does it revolve while the car runs 1 mi.?

6. The front wheel on a wagon is 3 ft. 9 in. in diameter, and the diameter of the rear wheel is 4 ft. 8 in. How many more revolutions than the rear wheel does the front wheel make in going 1 mi.?

7. The diameter of a circle is 4 yd. Find its area in square feet.

8. The area of a circle is 64 π sq. yd. Find its radius and circumference.

9. A horse is tied in a pasture by a rope 40 ft. long. Over how many square rods can he graze?

10. The area of a circle is 4 π acres. What is its radius in feet?

11. A circular pond 100 ft. in diameter is surrounded by a walk 6 ft. wide. Find the area of the walk.

12. The diameter of a circle is 25 ft. Find the side of a square having the same area.

13. A semicircle and a circle have the same area. Find the ratio of their diameters.

14. The radii of two circles are 10 ft. and 26 ft. respectively. Find the radius of a circle having an area equal to the sum of the areas of the first two circles.

15. A circular race track is 1 mi. long. Find its diameter and the inclosed area in acres.

16. A hay barn 50 ft. by 75 ft. stands in a meadow. A horse is tied to one corner by a rope 90 ft. long. Find the area over which he can graze.

17. Leadville, Colorado, is 29° 30′ west of Washington and in the same latitude. How far is it from Leadville to Washington if 1° of longitude in this case is 53.6 mi.?

18. Jacksonville, Florida, and Austin, Texas, are in the same latitude. The longitude of Jacksonville is 5° west of Washington and that of Austin is about 25° 40′ west of Washington. Find the distance between these two cities if a degree of longitude at this latitude is 59.6 mi.

19. Two places on the equator are 21° 10′ east of Washington and 14° 22′ west of it respectively. Find the distance between them if a degree at the equator is $69\frac{1}{6}$ mi.

20. The moon goes round the earth in 27 da. 7 hr. 43 min. 12 sec. Show that it moves over 13° 11′ of its orbit in one day.

21. The radius of a circle is 4 ft. Find the radius of a circle whose area is nine times as great.

22. The radii of two circles are 3 and 12 respectively. Find the ratio of their areas.

23. The areas of two circles are in the ratio of 16 to 100. Find the ratio of their circumferences.

24. The earth rotates on its axis once in about 24 hr. How far will a point on the equator move around the earth's axis in 1 min.? ($R = 3960$ mi.)

25. The earth's distance from the sun is 92,870,000 mi. Regarding the earth's orbit as circular, show that the earth travels about $18\frac{1}{2}$ mi. in 1 sec., using $365\frac{1}{4}$ da. to a year.

Summary

(Black figures refer to sections)

389. Résumé of Book V. 1. *Definitions.* Regular polygons, **350**; center of regular polygon, **356**; radius of regular polygon, **357**; apothem of regular polygon, **358**; angle at center of regular polygon, **359**; circumference of circle and polygon, **368**; meaning of π, **371**; mean and extreme ratio, **383**.

2. *Constructions.* To inscribe a square in a circle, **353**; to inscribe a regular hexagon in a circle, **354**; division of a line internally in mean and extreme ratio, **385**; inscription of a regular decagon in a circle, **386**; inscription of regular pentagon, **387**; inscription of regular polygon of 15 sides, **388**.

3. *Theorems on regular polygons.* An equilateral polygon inscribed in a circle, **352**; a circle can be circumscribed about or inscribed in any regular polygon, **355, 360**; two regular polygons each having n sides are similar, **364**; perimeters of two regular polygons of the same number of sides are in the same ratio as their sides, radii, or apothems, **365**; the area of a regular polygon is half the product of its perimeter and its apothem, **366**; the areas of two regular polygons of n sides have the same ratio as the squares of their radii or their apothems, **367**; if the sides of a regular inscribed polygon be repeatedly doubled, the perimeters of the successive polygons approach the circumference of the circle as a limit and the apothem approaches the radius, **369**; the number of inscribed regular polygons which have an odd number of sides is not known (page 394).

4. *Problems.* Computing the value of one side of a regular polygon in terms of the radius and the side of a regular polygon of half as many sides, **372**; computation of the

value of π, 373; computation of the numeric values of the segments of a line divided internally in mean and extreme ratio, 384.

5. *The circle.* Relation between the circumference of a circle and the perimeters of the inscribed and circumscribed polygons as the number of sides of each increases, 369; $C \div D = \pi$, 371; $C = 2\ \pi R$, 371; area of a circle, 375, 376, 377, 378; area of a sector, 379; area of a segment, 380; area of segment by trigonometry, 381; approximate formula for area of a segment, 382.

THE NUMBER π

No number is more important in mathematics and its applications than the number π. The significant rôle it has played in the development of mathematics can merely be hinted at in a short note, for the history of π runs back over four thousand years. Early geometers perceived that the area of a circle is $\frac{rC}{2}$. They saw that a square whose side was a mean proportional between C and $\frac{r}{2}$ would have the same area as the circle, and called the problem, the solution of which required the finding of C, "squaring the circle." This is essentially the problem of determining the value of π, and it was from the considerations just stated that the attempts of mathematicians to determine π by the ruler and compasses began.

The earliest reference to π is by an Egyptian priest, Ahmes (1700 B.C.). He gives $\pi = 3.1604$, which is the value used by the Egyptians.

The value 3 for π is mentioned in the Bible in two places: 1 Kings vii, 23; and 2 Chronicles iv, 2. This value was probably derived from the Babylonians.

Archimedes (287–212 B.C.) showed that the value of π lay between $3\frac{1}{7}$ and $3\frac{10}{71}$. The astronomer Ptolemy (about A.D. 150) gave $\pi = 3\frac{17}{120} = 3.14166$, a value more nearly correct than that of Archimedes.

The Hindus had various values for π. Aryabhata (about A.D. 510) gave $\pi = 3.1416$, a value criticized by later Hindu writers.

The Chinese at an earlier date than the Hindus worked out accurate values of π. Ch'ang Höng (about A.D. 125) gave $\pi = \sqrt{10} = 3.1623$. Tsu Ch'ung-chih (about A.D. 470), using a ten-foot circle, obtained a value of π between 3.1415927 and 3.1415926, which is correct to five decimals. He inferred, incorrectly, that these two values were equal to $\frac{3\,5\,5}{1\,1\,3}$ and $\frac{2\,2}{7}$ respectively. About the year 1600 Adriaen Anthoniszoon, a European mathematician, independently obtained the first of these values, $\pi = \frac{3\,5\,5}{1\,1\,3}$. About the same time another computer obtained the value of π to 17 decimals, and still another a value to 35 decimals.

With Newton's invention of calculus easier methods of obtaining π than the use of inscribed polygons were made possible. A simple application of calculus shows that

$$\pi = 4(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} \cdots).$$

Still other series better adapted to computation were derived in this way. In 1789, using modern methods, Vega (1756–1802) carried the value of π to 136 places. Later Dase carried it to 200 places, Richter to 500 places, and Shanks, in 1853, to 707 places. Such excessively lengthy values of π are never needed. Newcomb the astronomer says, "Ten decimals are sufficient to give the circumference of the earth to a fraction of an inch."

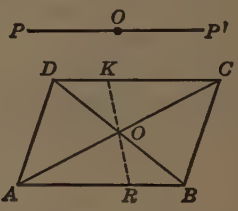
"Is π a rational number like $\frac{3\,5\,5}{1\,1\,3}$ or an irrational like $\sqrt{10}$?" This question interested the mathematicians greatly until 1794, when Legendre proved that π and π^2 were both irrational. But the question "Can π be constructed by ruler and compass?" still remained. Since $\sqrt{10} = \sqrt{2 \cdot 5}$, it is an irrational number which can be constructed by finding a mean proportional between 2 and 5. It can be seen, therefore, that this last question is more subtle than the first one. This question was not answered until 1882, when Lindemann proved that π could not be constructed by ruler and compasses.

Supplementary Topics

390. Symmetry. The beauty and attractiveness of many figures is dependent on the property of *symmetry*. This property is exhibited by snowflakes, flowers, butterflies, and many other natural objects. In artistic designs symmetry is often an essential property. The property of symmetry may exist with respect either to some point of the figure or to some straight line in it.

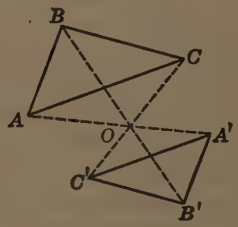
391. Symmetry with respect to a point. Points P and P' are *symmetric with respect to* O if O is the mid-point of PP'.

A figure is symmetric with respect to a point O if for each point of the figure there is a corresponding point such that the two points are symmetric with respect to O.



Thus, a parallelogram is symmetric with respect to the point of intersection of its diagonals.

Two separated figures may be symmetric with respect to a point if corresponding distances from that point are proportional.



Thus, $\triangle ABC$ and $\triangle A'B'C'$ are symmetric with respect to the point O, but O does not bisect AA', BB', or CC', nor is $\triangle ABC \cong \triangle A'B'C'$.

EXERCISES

1. Is a rhombus symmetric with respect to any point? What point?

2. Is there any triangle that is symmetric with respect to a point? What triangle? What point?

3. Are any parallelograms symmetric with respect to a point? What point?

4. Is a regular hexagon symmetric with respect to a point? a regular pentagon?

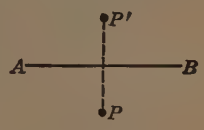
5. If a regular polygon has an even number of sides, is it symmetric with respect to a point?

6. Change the word "even" to "odd" in the preceding question, and give the answer.

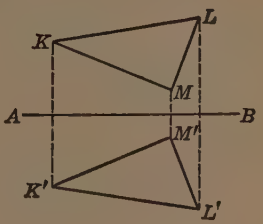
7. Are your two hands symmetric in any position? What position? Are your two feet symmetric?

8. What kind of trapezoid has a center of symmetry?

392. Symmetry with respect to an axis. Two points, P and P', are *symmetric with respect to an axis* AB if it is the perpendicular bisector of the line joining the two points.



Two figures are symmetric with respect to an axis AB if for every point K in one figure there is a corresponding point K' in the other such that the two points are symmetric with respect to the axis.



EXERCISES

1. Has a regular hexagon an axis of symmetry? a regular pentagon?

2. What triangle has an axis of symmetry? What is the axis? Is there more than one?

3. How many axes of symmetry has a regular hexagon? a regular pentagon?

4. How many axes of symmetry has a regular polygon of n sides if n is odd? How many if n is even?

5. Has a circle an axis of symmetry? more than one? Has a circle a center of symmetry?

6. What kind of symmetry has a butterfly? a house fly? a bee?

7. What kind of symmetry has a three-leaved clover?

8. What kind of trapezoid has an axis of symmetry? Locate it.

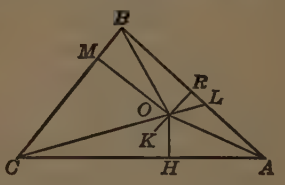
9. What kind of symmetry has an airplane?

10. Is there any difference in the symmetry of a biplane and a monoplane?

11. What letters of the alphabet are symmetric?

12. Which have point symmetry? axial symmetry?

393. Fallacies. Many fallacious geometric "proofs" exist in which the error is fairly well concealed. A few examples follow. Point out the error in these theorems:



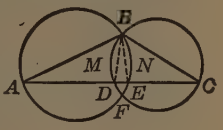
1. *Theorem. Any triangle is isosceles.*

Proof. Let $\triangle ABC$ be any triangle. Draw CL, the bisector of $\angle ACB$, and let KR be $\perp$ to AB at its mid-point R. Call the intersection of these two lines O. Then draw OH and $OM \perp$ respectively to AC and BC.

Then	$\triangle OCM \cong \triangle OCH$.		§ 88
Therefore	$HO = OM$,		
and	$HC = MC$.	(1)	§ 64
Now	$AO = BO$.		§ 136
Therefore	$\triangle AOH \cong \triangle BOM$.		§ 109
Therefore	$AH = BM$.	(2)	§ 64
(1) + (2),	$AH + HC = BM + MC$, or $AC = BC$.		

Therefore any triangle is isosceles.

2. ***Theorem.*** *There are two perpendiculars from a point to a line.*

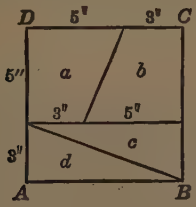


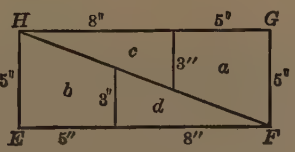
Proof. Let two circles, M and N, intersect in B and F. Draw the diameters AB and BC. Let AC cut the circles in D and E respectively. Then draw BD and BE.

$\angle AEB = 90°$, and $\angle BDC = 90°$. § 230

$\therefore BD$ and BE are both $\perp$ to AC.

3. If an 8-inch square is cut as indicated in $ABCD$ and the parts arranged as in $EFGH$, it may be made to *appear* that the areas of the two figures are equal, or that 64 sq. in. equals 65 sq. in.

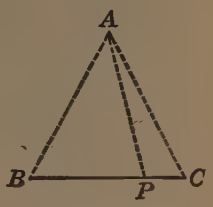




4. ***Theorem.*** *Any point on a line bisects it.*

Proof. In the figure below let BC be any line and P any point on it.

Construct an isosceles $\triangle ABC$ upon BC as base, and draw AP.



Since $\angle B = \angle C$, § 67

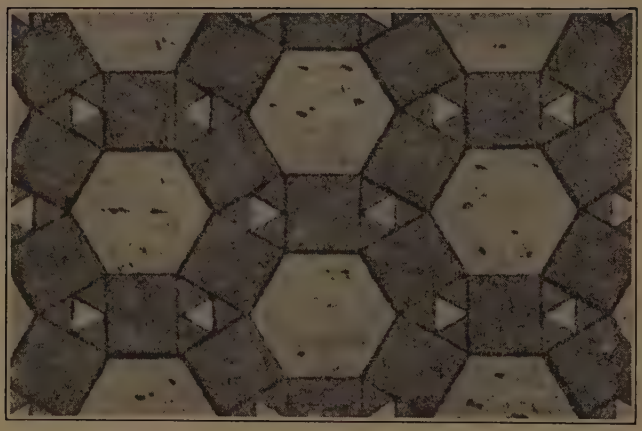
$AB = AC$, § 54

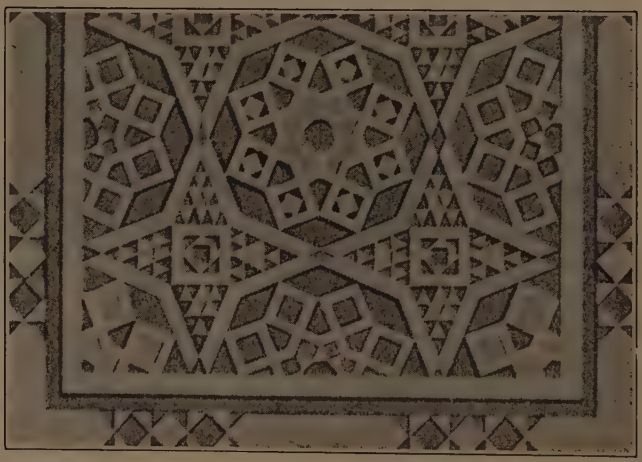
and $AP = AP$,

then $\triangle ABP$ is congruent to $\triangle ACP$.

$\therefore BP = PC$,

or any point on a line bisects it.



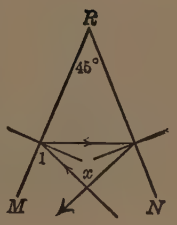


These pictures give examples of Byzantine art. The intricate designs employ regular polygons of three, four, six, and eight sides

APPLICATIONS

Book I

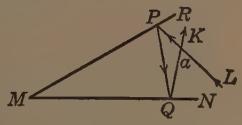
1. In the adjacent figure the angle between the two mirrors is 45°, and the ray of light goes in the direction indicated by the arrows. Find the number of degrees in $\angle x$ if (*a*) $\angle 1 = 70°$; (*b*) $\angle 1 = m°$.



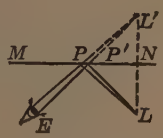
2. MR and MN are two mirrors, and $LPQK$ is the path of a ray of light which has been reflected at P and again at Q.

Prove $\angle a = 2 \angle M$.

If $\angle M$ is increased 5°, what change does this produce in $\angle a$?



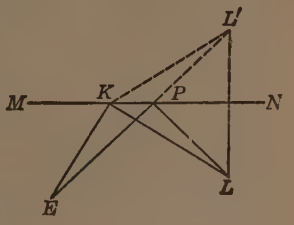
3. Two rays of light from L to points P and P' on the mirror may both enter the eye if P is near P'. The eye judges that L is in the line EP and also in the line EP'; that is, it appears to be at L' behind the mirror, where EP intersects EP'.



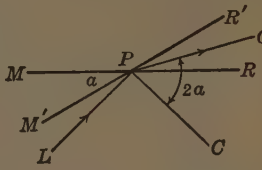
Prove LL' is $\perp$ to MN and that the image L' appears to be as far behind the mirror as L is in front of it.

4. The rays of light from L reflected from the mirror MN to the eye at E follow the path LPE.

Prove that the path LPE is shorter than any other path LKE.



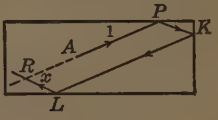
5. A ray of light LP is reflected from the mirror MR along the path PC. If the mirror swings through an angle a about P to the position $M'R'$, then the ray of light will be reflected along PC' so that $\angle CPC' = 2\,a$. Prove.



A billiard ball on striking a cushion without spinning is reflected as is a ray of light from a mirror, so that "the angle of incidence equals the angle of reflection," where these angles are measured between the path of the ball and a line perpendicular to the reflecting surface.

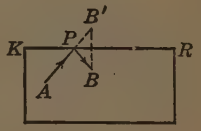
6. A billiard ball at A is struck so as to follow the path marked by the arrows. Find $\angle x$ if

(a) $\angle 1 = 42°$; (b) $\angle 1 = m°$.

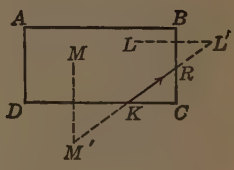


7. Is $PKLR$ in the figure of Exercise 6 a parallelogram? Prove.

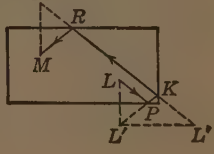
8. A billiard ball at A is struck so that after being reflected from KR it passes through B. How is B' located? Show that the line AB' determines the point of reflection P.



9. A billiard ball at M is struck so that after being reflected from DC and BC it strikes a ball at L. How are L' and M' located? Show that K and R are correctly determined by the line $L'M'$.

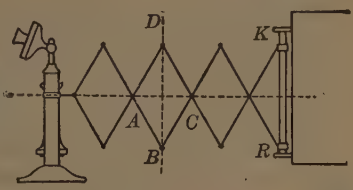


10. A billiard ball at L is struck so that after being reflected from the sides of the table it strikes the ball M. Show how points P, K, and R are determined.

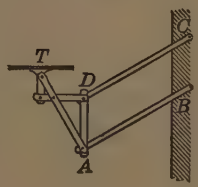


11. When this telephone bracket is pulled out, does AC remain horizontal? Why?

HINT. What kind of figure is $ABCD$? What angle does AC make with BD? What angle does AC make with KR? Is either K or R free to move?



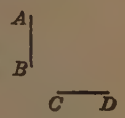
12. The corners of $\square ABCD$ are held tightly (but not rigidly) by friction pins, so that point A can be raised or lowered. Show that the table T will remain horizontal as A moves up and down.



BOOK II

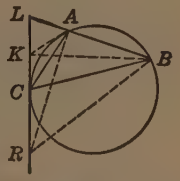
1. How can a carpenter's square be used to test whether a groove is semicircular or not?

2. AB and CD are the outer edges of two sidewalks which meet at right angles. The corner is to be rounded off by a 10-foot circle. Locate the center of the circle.



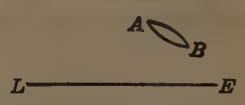
3. If AB and CD meet at an angle of less than 90°, and the sidewalk is to be rounded off as before, locate the center of the circle.

4. LAB is a secant and LR a tangent to a circle, the point of tangency being C. If angles are drawn with their vertices on LR and with their sides passing through A and B, prove that $\angle ACB$ is the greatest angle.



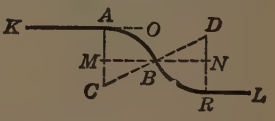
5. If, in the figure of Exercise 4, AB were a target and the marksman were compelled to fire at it from some point on line LR, what point would be best? Why?

6. AB is an old ship used for a target. The battleships move forward in the line LE. What point on LE is most favorable for a hit? How can this point be located?



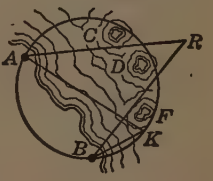
HINT. The geometric problem here is to pass a circle through two given points and tangent to a given line.

7. Parallel tracks KA and RL are to be connected by two equal circular arcs (a compound curve) of radius r so that arc AB is tangent to KA, arc BR is tangent to RL, and so that both have a common tangent, at B. If point A is fixed in advance and r is given, how may C, B, and D be determined and the arcs AB and BR drawn?

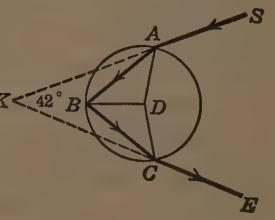


8. If, in Exercise 7, the point B and the direction of the common tangent BO are fixed in advance, how can C and the radius BC of the curve be constructed?

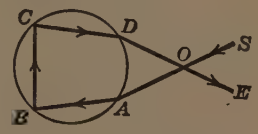
9. In the adjacent figure, A and B are two lighthouses and C, D, and F are rocks. Show that a ship will pass safely if it stays on a course such that $\angle R$ is less than $\angle K$.



10. The circle ABC is a cross section of a spherical drop of water. A ray of light SA entering the drop at A follows the path $SABCE$, being bent out of its path at A and at C. It is reflected from the inner surface of the drop at B so as to make equal angles with the radius to B. The angle between EC and SA is 42°, and $\angle KAD = 59°$. Show that $\angle ADB = 100°$, $\angle BAD = 40°$, and $\angle BAK = 19°$.



11. A ray of light SA may enter a water drop at A and follow the path $SABCDE$, being bent out of its path 19° on entering at A and leaving at D. At B and C it is reflected from the inner surface of the drop so as to make equal angles with the tangents at B and C. The rays cross at O, making an angle of 51°. Find the number of degrees in $\angle B$ and $\angle C$ and in the arcs AB, BC, CD, and DA.



When two rainbows appear in the sky at the same time the inner one is called the *primary* and the outer one the *secondary*. The conditions stated in Exercise 10 are essential in explaining the primary rainbow and those of Exercise 11 are essential in explaining the secondary rainbow.

12. Solve Exercise 11, using letters to represent the angles. Then from the results obtained in Exercise 11 show that $\angle O$ must equal the value given.

Book III

1. A map is drawn to a scale of 250 mi. to the inch. How far apart are two cities if the distance between them on the map is $2\frac{3}{8}$ in.?

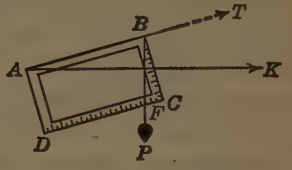
2. Two cities are 325 mi. apart. How far apart are they on a map drawn to a scale of 80 mi. to the inch?

3. The shadow of a tree is 42 ft. at the same time that the shadow of a 9-foot lamp post is 6 ft. long. Find the height of the tree.

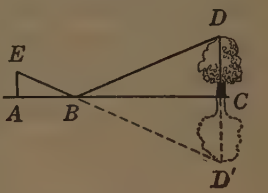
4. Given a trapezoid $ABCD$ and L at the mid-point of the base AB. Lines CL and DL produced intersect DA and CB produced at R and K respectively.

Prove that $\triangle DCK$ is equal in area to $\triangle DCR$.

5. The instrument of the adjacent figure is a simple form of height-measurer. The observer sights the object along AB and notes the point F where the line of the plumb bob crosses CD. If AB and AK are continued to the vertical line through the top of the object, the triangle thus formed will be similar to $\triangle BCF$. Then if BC and CF are known, and if the horizontal distance AK can be measured, prove that the height of the object above A can be computed.

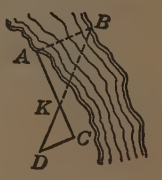


6. A horizontal mirror B on the ground reflects the image of the top of a tree to the eye of an observer at E. What distances in the figure, if measured, can be used to compute the height, CD, of the tree?



7. An observer at E sees the reflected image of the top of a tree formed by the water of a lake. If $AE = 5'$, $AB = 13'$, and $AC = 53'$, find CD.

8. The width AB of a river can be determined by similar triangles as follows. Lay out AC perpendicular to AB, and CD perpendicular to CA. Locate point K by sighting from D to B. The measurement of AK, KC, and CD will make it possible to compute AB. If $AK = 80'$, $KC = 25'$, and $CD = 40'$, find AB.



9. The top of a mountain can be seen 150 mi. out at sea. How high is the mountain? (Radius of earth is 3960 mi.)

HINT. $(150)^2 = x(x + 7920) = x^2 + 7920x$. (§ 294.) Neglect x^2 and solve the equation $7920x = (150)^2$.

10. A mountain on the seashore has a beacon light on its top 1400 ft. above the ocean. How far is the light visible at sea?

11. If h is the height in feet of an observer above the surface of the ocean and d is the distance in miles to an object on its surface just visible on the edge of the horizon, show that $d = \sqrt{\frac{3h}{2}}$, approximately.

12. From the bridge of a steamer 96 ft. above the sea a mountain known to be 1216 ft. above the sea is just visible on the horizon. Find by the formula of Exercise 11 the distance between the steamer and the mountain.

13. At what height must an observer in an airplane be in order to see over level country 50 mi. in all directions?

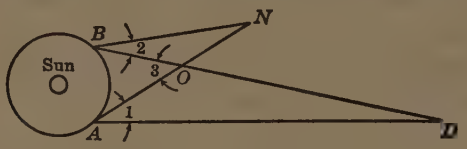
14. On the smooth ice of a lake three stakes of the same height are set up in a straight line, the distance between them being $\frac{3}{4}$ mi. An observer looking across the top of the first and third stakes finds that the line of sight is $4\frac{1}{2}$ in. below the top of the second stake. Determine from these results the diameter of the earth. (See Exercise 29, page 292.)

15. Letter paper of a certain style 6 in. long is of such a width that when folded once it is the same shape as the original sheet. Find the width.

16. The goal posts of a football field are on the goal line $18\frac{1}{2}$ ft. apart. During play the ball is carried across the goal line 24 ft. from the nearer post. If the ball is carried straight back from the point of crossing in order to kick goal, show that 32 ft. from the goal line is the best point from which to try for a goal. (See Exercise 4, page 409.)

17. A wireless mast 150 ft. high stands on a hill. From the base of the hill, lines to the top and the foot of the mast make angles of 60° and 45° respectively with the horizontal. Show that the height of the hill above the point of observation is $75(1 + \sqrt{3})$ ft.

18. The distance to a star was first measured in 1838 by Bessel. The following method is the simplest. Two stars near each other in the sky are selected, one, N, close to the earth, the other, D, vastly farther off. When the earth



is at A, $\angle 1$ is measured, and when it is at B, $\angle 2$ is measured. Then $\angle 3 = \angle D + \angle 1$ and $\angle 3 = \angle N + \angle 2$. Then $\angle N + \angle 2 = \angle D + \angle 1$, or $\angle N = \angle 1 - \angle 2 + \angle D$. But star D is so far off that $\angle D$ is small compared with $\angle N$, which is itself always less than one second of arc. Hence $\angle N = \angle 1 - \angle 2$, approximately. In $\triangle ABN$ all the angles and side AB are known. Show how to find AN.

Book IV

1. Find the hypotenuse of a right triangle if the other two sides are $2am$ and $a^2 - m^2$.

2. Find the sides of a right triangle if they are consecutive integers.

3. A flag rope is 3 ft. longer than the pole. When stretched out the rope reaches a point on the ground 21 ft. from the base of the pole. Find the height of the flagpole.

4. A flagpole 87 ft. high was broken by the wind. One end of the part broken off rested on the ground 60 ft. from the base of the pole and the other end remained fastened to the upright portion. How high above the ground did the break occur?

5. A boat travels north 52 mi., then east 26 mi., and finally north 25 mi. How far is it from the starting point?

6. A motor car travels east 45 mi., then north 30 mi., then west 6 mi., and finally north 59 mi. How far is it then from the starting point?

7. The stiffness of a beam of rectangular cross section varies as the product of the area of its cross section and the square of the height of the beam; that is, S varies as $(wh)h^2$, where w is the width of the beam and h is its height. Show that the stiffness of a beam 6″ by 9″ when placed so that the 6-inch dimension is horizontal is $2\frac{1}{4}$ times the stiffness when the 6-inch dimension is vertical.

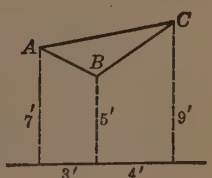
8. Compare the greatest stiffness of a beam 8″ by 12″ with that of a beam of the same material 9″ by 16″.

9. One dimension of the cross section of a beam is 12″, the other dimension is unknown. If the beam is 4 times as stiff when the 12-inch dimension is vertical as when it is horizontal, find the other dimension.

10. If a book is held 18 in. from a light and then moved so that it is 4 ft. distant, compare the brightness in the two cases.

11. The moon is 240,000 mi. away. If it were 60,000 mi. away, how would the brightness compare with what it is now?

12. Find the area of $\triangle ABC$ from the dimensions given in the adjacent figure.



Book V

1. Find the largest square beam which can be cut from a log 18 in. in diameter.

2. The diameter of the earth is 7920 mi. and that of the moon is 2160 mi. Find the length of their equators and the ratio of lengths.

3. The amount of water delivered by two pipes varies nearly as the area of cross section; that is, as the squares of their radii or of their diameters. If the water mains of a city are changed from 18 in. to 30 in., find the ratio of the old supply of water to the new.

4. A flywheel makes 1500 revolutions per minute. How long does it take to turn through 45°?

5. The regular equipment of a certain motor car is 32″ by 6″ balloon tires. These were temporarily replaced by 31″ by $5\frac{1}{2}$″ tires, and according to the meter a run of 840 mi. was made. Find the actual mileage.

6. The speedometer on a certain car was designed for use with 32″ by 6″ tires. The owner changed to 33″ by $6\frac{1}{2}$″ tires. At what speed was the car traveling when the meter registered 45 mi. per hour?

7. Two straight portions of a railroad track making an angle of 150° are connected by a circular arc whose radius is 720 ft. Find the length of the curved portion of the track.

8. The diameter of the piston of a steam engine is 24 in. If the steam pressure is 120 lb. per square inch, what is the total pressure on the piston?

9. A circular walk 10 ft. wide surrounds a garden 120 ft. in diameter. Find the area of the walk.

10. The diameter of a circle is divided into four equal parts, and semicircles are drawn as in the adjacent figure. Show that the circle is divided into four parts of equal area.

11. Semicircles are described on the radius of the larger circle of the adjacent figure.

Prove that the shaded area equals the area of the two small semicircles.

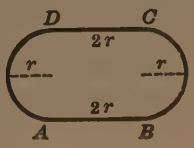
12. A belt runs over two wheels, one of which is 32 in. in diameter and the other 6 in. When the first wheel revolves 240 times per minute, how many revolutions does the second make in 5 seconds?

13. A 12-inch pipe is to be replaced by one which will deliver twice as much water. Find the diameter of the new pipe.

14. Two sewer pipes 30 in. in diameter and a third 40 in. in diameter empty into a fourth. If the latter is to carry the maximum flow of the first three, find the diameter.

15. A square piece of ground has a perimeter of 400 ft. Find the difference between the area of the square and the area of a circular piece having the same perimeter.

16. A quarter-mile track is made of two equal semicircular ends joined by two equal parallel sides. The diameter of the semicircular ends equals the length of one of the parallel sides. Find the dimensions of the track.



NON-EUCLIDEAN GEOMETRY

The postulate of § 82 is essentially the famous "parallel axiom" of Euclid. Many mathematicians have attempted to regard this postulate as a theorem and obtain a proof of it based on Euclid's other assumptions. Some — Legendre among them — thought they had obtained valid proofs, but all the attempts at proof, when carefully examined, were found to assume at some point the truth of the postulate itself. Indeed, the mere fact that Euclid was acute enough not to place this postulate with the others is regarded by some as enough evidence in itself to stamp him as a great mathematician.

At the present time no well-informed person attempts to prove this postulate. Correct notions in regard to it were arrived at independently by three men: Gauss (1777–1855), Bolyai (1802–1860), and Lobachevsky (1793–1856). Lobachevsky set himself, not the problem of proving it, but of finding out what would result if instead of it he assumed the following: "Through a point outside a line more than one parallel to the line can be drawn." Leaving the other assumptions of Euclid unchanged, he was then able to develop a perfectly consistent system of geometry very different from that of Euclid. His system is consistent because it leads to no contradictory conclusions — to no two theorems which make precisely opposite assertions. It does, however, lead to theorems which contradict theorems of Euclidean geometry. For example, one theorem of non-Euclidean geometry asserts that similar triangles do not exist. Another theorem asserts that the sum of the angles of a triangle is *less than* two right angles and the deficiency is proportional to the area of the triangle. Gauss arrived at this theorem and he attempted to prove it by taking three widely separated mountain stations as the vertices of a large triangle. Then as accurately as possible he measured each of its angles. The sum, however, came out so near a straight angle that the result proved nothing. Whatever the difference in the sum of the angles may be, it seems impos-

sible by measurements of the angles of any actual triangle to obtain its value.

Later Riemann replaced Euclid's axiom by the following: "Through a point outside a line no parallel to the line can be drawn." He then developed a system of geometry different from that of both Euclid and Lobachevsky.

Actually the three systems of geometry here referred to differ but little, and the system of Euclid is by far the most convenient. When, in 1905, Einstein began to formulate his theory of relativity, he based certain interesting developments of that theory on the non-Euclidean geometry of Riemann.

There is another type of non-Euclidean geometry, although it is not usually referred to under that name, which has received a great deal of attention for many years. This geometry has to do with the idea of dimensionality. In this text we have considered figures of only two dimensions. These dimensions we ordinarily refer to as length and breadth. Objects in the world about us have a third dimension, which we call thickness. As a continuation of the present text the study of Solid Geometry may be taken up, and the geometrical properties of figures in three dimensions may be studied. For example, corresponding to the result of Theorem 11 on page 384, which demonstrates that the area of a circle equals one half the product of the circumference and the radius, we find that the volume of a sphere is equal to one third the product of the surface of the sphere and its radius.

After one has passed from the study of geometry of two dimensions to that of three dimensions, the question arises as to the possibility of studying geometry of four or even more dimensions. The fact that stops us is that no one is able to picture the appearance of objects in more than three dimensions. We are apparently three-dimensional creatures, and even if there is a fourth dimension in which our three dimensions are immersed, just as a plane might be thought of as immersed in three-dimensional space, our senses are not so constituted as to get any response to stimuli that originate outside of our familiar world of three dimensions. It would account for many

of the vagaries of unsettled minds if some of their imaginings could be interpreted by the assumption that they know how to enter, either wholly or in part, into a dimension of space which the rest of us know nothing about.

Some mathematicians have reduced the discussion to a purely numerical or algebraic basis. The method of making diagrams on a two-dimensional piece of plotting paper to represent equations in two letters, x and y, is developed in elementary algebra. This method of picturing equations in two variables in two-dimensional space is taken up in an orderly manner in the subject of Analytical Geometry. Thus the equation $x^2 + y^2 = 1$ is represented by a circle of radius 1. If we try to represent graphically the equation $x^2 + y^2 + z^2 = 1$, we must have three dimensions to do so. As a matter of fact this equation in Solid Analytical Geometry is represented by a sphere of radius 1. But if we have the equation $x^2 + y^2 + z^2 + u^2 = 1$, it is obvious that representation of the same character would require four dimensions of space, which we do not have. Nevertheless such an equation is sometimes referred to as a sphere in four-dimensional space just as a manner of speech. Although we are unable to visualize a space of four dimensions, some scientists, notably Einstein, have interpreted equations in four letters, using our ordinary x, y, and z to denote three dimensions and u to denote time.

FORMULAS OF PLANE GEOMETRY

Notation

a, b, c = sides of triangle
c = hypotenuse of right triangle
h = altitude
b, b_1, b_2 = bases
m_c = median on side c
s = side of regular polygon
a = apothem of regular polygon
p = perimeter of regular polygon
R and r = radius of circumscribed circle
C = circumference of circle
D = diameter of circle
A = area

Lengths of Lines

1. Right triangle, $c = \sqrt{a^2 + b^2}$.
2. Equilateral triangle, $h = \frac{s}{2}\sqrt{3}$.
3. Inscribed equilateral triangle, $s = r\sqrt{3}$.
4. Circumscribed equilateral triangle, $s = 2r\sqrt{3}$.
5. Square, diagonal $= s\sqrt{2}$.
6. Inscribed square, $s = r\sqrt{2}$.
7. Circumscribed square, $s = 2r$.
8. Inscribed regular hexagon, $s = r$.
9. Circumscribed regular hexagon, $s = \frac{2r}{3}\sqrt{3}$.
10. Circle, $D = 2R$.
11. Circle, $C = 2\pi R = \pi D$.

12. Circle, arc $= \dfrac{\text{central angle}}{180^\circ}\,\pi R$.

13. Median of a triangle, $m_c = \sqrt{\dfrac{a^2+b^2}{2} - \dfrac{c^2}{4}}$.

14. Diameter of circumscribed circle of triangle,
$$D = a \cdot b \div h_c.$$

15. Bisector of $\angle C$ of $\triangle ABC = \dfrac{\sqrt{ab[(a+b)^2 - c^2]}}{a+b}$

Areas

16. Radius of inscribed circle of a triangle,
$$r = \frac{\text{area of triangle}}{\frac{1}{2}(a+b+c)}.$$

17. Parallelogram, $A = bh$.

18. Square, $A = s^2$.

19. Triangle, $A = \dfrac{bh}{2}$.

20. Triangle, $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$.

21. Equilateral triangle, $A = \dfrac{s^2}{4}\sqrt{3}$.

22. Trapezoid, $A = \dfrac{h(b_1+b_2)}{2}$.

23. Regular polygon, $A = \dfrac{ap}{2}$.

24. Circle, $A = \pi r^2$.

25. Circular ring, $A = \pi R^2 - \pi r^2 = \pi(R+r)(R-r)$.

26. Sector, $A = \dfrac{\text{central angle}}{360}\,\pi R^2$.

27. Segment, $A =$ sector $-$ triangle.

28. Segment, $A = \dfrac{h^3}{2b} + \dfrac{2bh}{3}$ (approximate)

($h =$ height of segment, $b =$ chord of segment).

TABLES

Length

English Units

12 inches (in.)	= 1 foot (ft.)
3 feet	= 1 yard (yd.)
$5\frac{1}{2}$ yards	= 1 rod (rd.)
320 rods	= 1 mile (mi.)

Metric Units

10 millimeters (mm.)	= 1 centimeter (cm.)
10 centimeters	= 1 decimeter (dm.)
10 decimeters	= 1 meter (m.)
10 meters	= 1 dekameter (Dm.)
10 dekameters	= 1 hektometer (hm.)
10 hektometers	= 1 kilometer (km.)

Equivalents

1 inch	= 2.54 centimeters	1 centimeter	= 0.3937 inch
1 foot	= 30.48 centimeters	1 meter	= 39.37 inches
1 mile	= 1.6093 kilometers	1 kilometer	= 0.6214 mile (mi.)

Square Measure

English Units

144 square inches (sq. in.)	= 1 square foot (sq. ft.)
9 square feet	= 1 square yard (sq. yd.)
$30\frac{1}{4}$ square yards	= 1 square rod (sq. rd.)
160 square rods	= 1 acre (A.)
640 acres	= 1 square mile (sq. mi.)

Powers and Roots

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.260	52	2,704	140,608	7.211	3.733
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.780
5	25	125	2.236	1.710	55	3,025	166,375	7.416	3.803
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.826
7	49	343	2.646	1.913	57	3,249	185,193	7.550	3.849
8	64	512	2.828	2.000	58	3,364	195,112	7.616	3.871
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.893
10	100	1,000	3.162	2.154	60	3,600	216,000	7.746	3.915
11	121	1,331	3.317	2.224	61	3,721	226,981	7.810	3.937
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.958
13	169	2,197	3.606	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.742	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.873	2.466	65	4,225	274,625	8.062	4.021
16	256	4,096	4.000	2.520	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.062
18	324	5,832	4.243	2.621	68	4,624	314,432	8.246	4.082
19	361	6,859	4.359	2.668	69	4,761	328,509	8.307	4.102
20	400	8,000	4.472	2.714	70	4,900	343,000	8.367	4.121
21	441	9,261	4.583	2.759	71	5,041	357,911	8.426	4.141
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.796	2.844	73	5,329	389,017	8.544	4.179
24	576	13,824	4.899	2.885	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.963	76	5,776	438,976	8.718	4.236
27	729	19,683	5.196	3.000	77	5,929	456,533	8.775	4.254
28	784	21,952	5.292	3.037	78	6,084	474,552	8.832	4.273
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.291
30	900	27,000	5.477	3.107	80	6,400	512,000	8.944	4.309
31	961	29,791	5.568	3.141	81	6,561	531,441	9.000	4.327
32	1,024	32,768	5.657	3.175	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.745	3.208	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.831	3.240	84	7,056	592,704	9.165	4.380
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.220	4.397
36	1,296	46,656	6.000	3.302	86	7,396	636,056	9.274	4.414
37	1,369	50,653	6.083	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.362	88	7,744	681,472	9.381	4.448
39	1,521	59,319	6.245	3.391	89	7,921	704,969	9.434	4.465
40	1,600	64,000	6.325	3.420	90	8,100	729,000	9.487	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.498
42	1,764	74,088	6.481	3.476	92	8,464	778,688	9.592	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.644	4.531
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.547
45	2,025	91,125	6.708	3.557	95	9,025	857,375	9.747	4.563
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.798	4.579
47	2,209	103,823	6.856	3.609	97	9,409	912,673	9.849	4.595
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.950	4.626
50	2,500	125,000	7.071	3.684	100	10,000	1,000,000	10.000	4.642

NEW-TYPE TEST QUESTIONS

In the following pages are given sets of test questions covering the entire five books of the New Plane Geometry. These test questions have been prepared in accordance with the most successful modern practice. They furnish to the teacher a valuable means for determining the knowledge of the student, and also supply the student with an outline of the material which he needs to know in order to have a thorough grasp of plane geometry.

Scoring the Tests

The students should not be allowed to mark in the book. A convenient way of giving these tests is to let the student arrange his responses in a vertical column on a separate sheet of paper. The responses should be numbered to correspond with the numbers of the questions. If the student marks "T" or "F" for each question of the "true-false" tests and places his name on the sheet, the scoring and identification of the tests will be much simplified.

Each question in these tests should be marked for correctness or incorrectness. The score for the "completion" questions is the number of questions answered correctly. The score for the "true-false" questions is the number answered correctly minus the number answered incorrectly.

It is suggested that in normal cases a fair passing mark might be placed at the upper limit of the poorest quarter of the class. This is only an estimate, and the teacher will be the best judge of the proper mark.

BOOK I. TEST I

Completions (I—C-1)

1. If a side and the two adjacent angles of one triangle are equal respectively to a side and the two adjacent angles of another, the triangles are _____.

2. The two acute angles of a right triangle are _____.

3. The line joining the mid-points of the nonparallel sides of a trapezoid is parallel to the bases and equal to _____.

4. The sum of the supplements of two supplementary angles is _____.

5. In an isosceles trapezoid whose base angles are 60° the upper base is 10 in. and each side is 5 in. The lower base is _____ in.

6. Of all parallelograms with base 5 in. and altitude 3 in. the one with the least perimeter has its other sides each equal to _____ in.

7. If the bisector of an exterior angle at the base of an isosceles triangle is parallel to one of the sides, the number of degrees in the vertex angle of the triangle is _____.

8. The sum of the exterior angles of a right triangle formed by the prolongations of the hypotenuse is _____.

True-False Questions (I—TF-1)

1. Two geometric figures which cannot be made to coincide are not congruent.

2. The altitude of a triangle divides it into two equal triangles.

3. If a line fails to intersect one of two parallel lines, it is parallel to the other.

4. The sum of the interior angles of a triangle is two straight angles.

5. A parallelogram is not a trapezoid.

6. If the sum of the interior angles of a polygon is three times the sum of the exterior angles, the polygon has six sides.

7. Two triangles are congruent if two sides and the median to one of them in one triangle are equal respectively to the corresponding lines in the other.

8. A line from a vertex of a triangle to a point in the opposite side is shorter than either of the other two sides.

9. If two right triangles are congruent, the sum of the sides of one exceeds the hypotenuse of the other.

10. Lines perpendicular to the sides of a right triangle intersect at right angles.

11. If the diagonals of a quadrilateral bisect each other at right angles, it is a square or a rhombus.

12. If the four sides of a quadrilateral are equal respectively to the four sides of a parallelogram, then the quadrilateral is itself a parallelogram.

13. Angles adjacent to the same angle or to equal angles are equal.

14. The median of a trapezoid bisects the bases.

15. Two polygons of seven sides each are congruent if six sides and six angles of one are equal respectively to the corresponding parts in the other.

16. The intersection of the three angle bisectors of a triangle is equally distant from the three vertices.

17. In a triangle the smallest angle lies opposite the shortest side.

18. The sum of the complements of the angles of a quadrilateral equals the sum of the exterior angles.

19. If the altitude and the median of a triangle from the same vertex have the same length, the triangle is isosceles (or equilateral).

20. Since an axiom does not require proof it does not have a hypothesis and a conclusion.

21. Through a given point not on a line there is only one line making an angle of 60° with the first.

22. If two angles have their sides parallel their bisectors are parallel.

23. The number of triangles which can be constructed having given two sides and an angle opposite one of them is unlimited.

24. If there exist two points on a surface such that the straight line joining them lies entirely on the surface, then the surface is a plane.

25. If a line from a vertex of a triangle to a point in the base divides it into two congruent triangles, the line is an altitude of the triangle.

TEST II

Completions (I—C-2)

1. If a triangle is isosceles, the angles opposite the equal sides are _____.

2. If two lines are cut by a transversal making a pair of corresponding angles equal, the two lines are_____.

3. If two sides of a triangle are unequal, the angles opposite them are _____, and the greater angle lies opposite the _____ side.

4. The altitude of a triangle bisects the base. One side (not the base) of the triangle is 6. The other side is _____.

5. Two sides of a triangle are 10 in. and 13 in. The third side exceeds _____ in.

6. The base of a triangle is 12 and the median to the base is 6. The sum of the two angles adjoining the base is _____.

7. If each exterior angle of a polygon of n sides is a degrees, the sum of the interior angles is _____.

8. One of the equal sides of an isosceles triangle is 6 in. If parallels to the sides are drawn from any point in the base, the perimeter of the parallelogram thus formed is _____ in.

True-False Questions (I—TF-2)

1. If two triangles have a common side, and the remaining sides of one are equal respectively to the remaining sides of the other, the triangles are congruent.

2. An angle which is equal to its supplement is a right angle.

3. If a polygon has n exterior angles, the polygon has $(n-2)$ sides.

4. The converse of a theorem is not true.

5. A triangle may have two equal sides though no two angles are equal.

6. The difference between the arms of a right triangle exceeds the hypotenuse.

7. If one line meets another so as to make two adjacent angles equal, the lines are perpendicular to each other.

8. If one of two parallel lines does not intersect a third line, then neither does the other.

9. If the sum of the interior angles of a polygon is 4 right angles. the polygon is a parallelogram.

10. Two isosceles triangles are congruent if a side and a base angle of one are equal to a side and a base angle of the other.

11. A corollary is a statement which is accepted as true without proof.

12. If two angles have their sides parallel each to each, they are equal.

13. In an equiangular triangle the line joining the mid-points of two sides is equal to one sixth of the perimeter.

14. The angle formed by the base of an isosceles triangle and the altitude upon one side is one half of the vertex angle.

15. If two congruent triangles have an angle of one equal to an angle of the other, the sides opposite these angles are equal.

16. If two angles of a triangle are the complements respectively of two angles of another triangle, then the third angles are supplementary.

17. Vertical angles are never supplementary.

18. Any line must meet at least two of the sides, or the prolongations of the sides, of any triangle.

19. If two straight lines are cut by a transversal making a pair of alternate interior angles unequal, the lines are parallel.

20. The perpendicular bisectors of two sides of a triangle intersect at the mid-point of the third.

21. Two parallelograms are congruent if two adjacent sides and the included angle of one are equal to the corresponding parts in the other.

22. If two opposite angles of a quadrilateral are bisected by a diagonal, the quadrilateral is a parallelogram.

23. If a line from the vertex of a triangle to a point in the base divides the triangle into two congruent triangles, the triangle must be isosceles.

24. If the altitudes, the medians, and the angle bisectors of a triangle all intersect at the same point, the triangle is equilateral.

25. The angle between the bisectors of two consecutive angles of a quadrilateral is one half of the difference of the other two angles.

TEST III

Completions (I — C-3)

1. If the three sides of one triangle are equal respectively to the three sides of another, the triangles are _____.

2. The opposite sides of a parallelogram are _____ in length.

3. The sum of the angles of a triangle is equal to _____ degrees.

4. If one diagonal of a rhombus is equal to a side, the number of degrees in each acute angle of the rhombus is _____.

5. If a side of an isosceles triangle is 10 in., a line drawn from the mid-point of one side to the mid-point of the base has a length of _____ in.

6. The sum of three interior angles of a quadrilateral diminished by the fourth exterior angle is _____ degrees.

7. The angle bisectors of an equiangular triangle intersect at a point 3 in. from each side. The length of a median is _____ in.

8. One angle of an isosceles trapezoid is 110°. The other angles are _____ and _____.

True-False Questions (I — TF-3)

1. In an equilateral triangle each angle is 60°.

2. The altitude to a side of a triangle bisects the angle opposite the side.

3. If two lines are drawn at random, either they are parallel or they meet at one and only one point.

4. The complement of an acute angle is obtuse.

5. A diagonal of a rectangle divides it into two congruent right triangles.

6. The sum of two angles of a triangle is greater than the third.

7. An equiangular polygon is equilateral.

8. Two parallelograms are congruent if the four sides of one are equal respectively to the four sides of the other.

9. The sum of the altitudes of a triangle is at least as great as the sum of the angle bisectors.

10. An angle of a quadrilateral is less than 90°.

11. The diagonals of a parallelogram bisect its angles.

12. The sum of the diagonals of a quadrilateral exceeds the perimeter.

13. In every polygon each side is less than the sum of all the other sides.

14. If a point lies within an angle, a line can always be constructed through the point such that the segment included between the two sides of the angle is bisected by the point.

15. Two altitudes of a triangle cannot have more than one point in common.

16. The line segments joining the mid-points of the opposite sides of an isosceles trapezoid bisect each other at right angles.

17. If the sum of two exterior angles of a triangle equals three right angles, the triangle is a right triangle.

18. Two triangles are congruent if two angles and a side of one are equal to two angles and a side of the other.

19. Two triangles are congruent if two sides and the line segment joining the mid-points of these sides in one are equal to the corresponding parts in the other.

20. The product of the sides of a right triangle exceeds the square of the hypotenuse.

21. Each angle of an equilateral hexagon is 120°.

22. Two rhombuses are congruent if their diagonals are equal.

23. Figures congruent to the same or to congruent figures are congruent.

24. A convex polygon can have more than three acute angles.

25. If the mid-point of one side of a triangle is equally distant from the three vertices, the triangle is a right triangle.

BOOK II. TEST I

Completions (II — C-1)

1. In the same circle or in equal circles, if two central angles are equal, their intercepted arcs are _____.

2. In the same circle or in equal circles, if two chords are unequal, the greater chord has the _____ minor arc.

3. If two circles are tangent externally or internally at point A, the line of centers passes through _____.

4. The maximum number of circles determined by 4 points is _____.

5. If two tangents of a circle intersect at right angles, the greater arc contains _____ degrees.

6. Three circles are tangent externally, each to the other two. The lines of centers are 12 in., 16 in., and 18 in. The radii are _____ in., _____ in., and _____ in.

7. A chord in a circle of radius 6 in. makes an angle of 30° with a tangent. The length of the chord is _____ in.

8. Side AB of an inscribed triangle ABC subtends an arc of 100°. The sum of angles A and B is _____ degrees.

True-False Questions (II — TF-1)

1. A line which bisects a circle is double the radius.

2. In the same or equal circles, if two chords are unequal, the greater chord has the greater major arc.

3. The vertices of a circumscribed polygon lie on the circle.

4. If two chords make equal angles with a third chord (not a diameter) at opposite extremities of this chord, then they are unequal.

5. An angle inscribed in a circle is smaller than an angle whose vertex is inside the circle and which intercepts an equal arc.

6. A point is within, on, or outside a circle according as its distance from the center is greater than, equal to, or less than the radius.

7. If two arcs in two circles have the same length, they contain the same number of degrees.

8. If two triangles are congruent, their circumscribed circles are congruent.

9. In the same or equal circles, the smaller central angle intercepts the smaller minor arc.

10. There cannot be more than one line tangent to a circle at a given point on the circle.

11. All circles tangent to a given line at a given point have their centers on a line perpendicular to the given line at that point.

12. A circle can always be inscribed in any polygon.

13. If two secants of a given circle cut off equal arcs between them, they are parallel.

14. If two circles have one point in common, they also have a second point in common.

15. If a straight line bisects a chord of a circle, it bisects the arc subtended by the chord.

16. If one chord of a circle is twice the length of another, the shorter is twice as far from the center as the longer.

17. In two unequal circles equal lengths of arc subtend unequal central angles.

18. The sum of the sides of a right triangle circumscribed about a circle is equal to the sum of its hypotenuse and the diameter of the circle.

19. The radius of a circle circumscribed about an equilateral triangle is two thirds the altitude.

20. A single point cannot be a locus.

21. In three equal circles equal arcs are subtended by equal chords.

22. A circle can always be constructed tangent to two given lines.

23. A circle can be inscribed in any rectangle.

24. The shortest distance to a circle from a point inside is never greater than a radius.

25. The bisectors of all the angles inscribed in a given arc of a circle meet at a common point on the circle.

TEST II

Completions (II — C-2)

1. In the same circle or in equal circles, if two arcs are equal, their central angles are _____.

2. In the same circle or in equal circles, if two minor arcs are unequal, the greater arc has the _____ chord.

3. If two circles intersect, the line of centers bisects their _____ at an angle of _____ degrees.

4. In a circle of radius 5 in. a chord of length 5 in. subtends a central angle of _____ degrees.

5. Two tangents to a circle of radius 5 in. intersect at an angle of 60°. The longest secant which can be drawn from the common point has a length of _____ in.

6. Two chords are perpendicular to a third at its extremities. The length of one of the two chords is 6 in. The length of the other chord is _____ in.

7. An angle of 20° is formed by a secant and a tangent which meet outside a circle. The greater of the intercepted arcs is 90°. The angles of the triangle formed by the three points on the circumference are _____, _____, and _____ degrees.

8. If a point is equidistant from the sides of a triangle, it is the point of intersection of the _____ of the triangle.

True-False Questions (II — TF-2)

1. An arc intercepted by a central angle of 60° is less than a radius in length.

2. The common chord of two intersecting circles cannot be tangent to either circle.

3. A rhombus can always be circumscribed about a given circle.

4. If two chords intersect at right angles, the four arcs of the circle are equal.

5. In unequal circles equal central angles intercept unequal arcs.

6. If a right triangle be revolved in its plane about the vertex of the right angle, the other vertices describe two circles of unequal radii.

7. If two triangles are inscribed in congruent circles, they are congruent.

8. If a central angle in one circle exceeds that in another, the first intercepted arc exceeds the second in length.

9. Two distinct circles which have no common tangent line cannot be congruent.

10. If a circle can be inscribed in a parallelogram, the parallelogram is a rhombus.

11. If two unequal circles are tangent externally and tangents are drawn to the circles from a point in the common tangent line, the tangent to the smaller circle is longer than that to the greater.

12. If the length of one arc of a given circle is twice that of another, the central angle intercepted by the first is twice the central angle intercepted by the second.

13. A chord can always be found which passes through a given point inside a circle and which is bisected by the point.

14. In order that a quadrilateral may be inscribed in a circle it is necessary that the opposite angles be supplementary.

15. The angle between two intersecting secants is measured by one half of the difference in degrees of the intercepted arcs.

16. Corresponding segments of equal intersecting chords are equal.

17. Any point which satisfies required conditions is a locus.

18. A circle cannot always be constructed tangent to each of four given lines.

19. If two circles are concentric, chords of the larger which are not tangent to the smaller are unequal.

20. If two intersecting chords divide a circle into two pairs of equal arcs, both chords are diameters.

21. The extremities of two equal chords are the vertices of a trapezoid.

22. A common external tangent of two circles is greater than the line of centers.

23. If the perpendiculars from the center upon the sides of an inscribed polygon are equal, the polygon is equiangular.

24. A circle constructed on one of the equal sides of an isosceles triangle as diameter bisects the base.

25. Two chords which make equal angles with a third at its opposite extremities are equal.

TEST III

Completions (II—C-3)

1. In the same circle or in equal circles, if two chords are equal, their subtended arcs are _____.

2. In the same circle or in equal circles, if two chords are unequal, the greater is at the _____ distance from the center.

3. If two tangents are drawn to a circle from an outside point, they are _____.

4. In a circle of radius 5 in. the length of the greatest chord is _____ in.

5. A chord of length equal to a radius is extended to meet a tangent at right angles. The number of degrees in the larger arc intercepted between the tangent and the secant is _____.

6. A chord is half a radius distant from the center. Tangents drawn at the ends of the chord intersect at an angle of _____ degrees.

7. If the four vertices of a trapezoid lie on a circle, the trapezoid is _____.

8. A circle is inscribed in a triangle whose sides are 9 in., 8 in., and 9 in. The segments of the equal sides are _____ in. and _____ in.

True-False Questions (II—TF-3)

1. Any major arc of a given circle exceeds any minor arc.

2. The perpendicular bisector of a chord passes through the center of the circle, except when the chord is a diameter.

3. Two chords of a circle cannot be equal unless their central angles are equal.

4. If there is more than one tangent to a circle through a point, the distance of the point from the center of the circle exceeds a radius.

5. Inscribed angles whose sides intersect on the circle are equal.

6. If a line meets a circle at all, it meets it in two points.

7. If the two arcs of a circle subtended by a chord are equal, then no other chord of the circle can exceed this one in length.

8. If two circles have one common tangent line, they have another.

9. A circle whose center is at a distance less than r from the center of a circle of radius r will intersect that circle.

10. If a circle can be inscribed in a quadrilateral, the quadrilateral is a parallelogram.

11. A line perpendicular to a radius meets the tangent at the extremity of this radius at some point outside the circle.

12. A quadrilateral inscribed in a circle is a parallelogram.

13. The radius of a circle circumscribed about a right triangle equals half the hypotenuse.

14. If two circles are tangent externally, their only common tangent is the perpendicular bisector of their line of centers.

15. If one chord of a circle is twice another, the arc subtended by the longer is greater than that subtended by the shorter.

16. The radius of a circle inscribed in an equilateral triangle is one third of the altitude.

17. The locus of the mid-points of a set of parallel chords is a diameter.

18. If two circles are tangent to each other, the distance between their centers is the sum of their radii.

19. Since a circle can always be inscribed in a triangle, a circle can be constructed tangent to each of three given lines.

20. If two unequal circles are concentric, chords of the larger which are tangent to the smaller are equal.

21. If through a point there is one tangent to a circle, there is also another.

22. If two circles are tangent externally at B, and AC is a common external tangent, then angle ABC is a right angle.

23. If one of two equal circles passes through the center of the other, tangents to the two circles at a common point are perpendicular to each other.

24. If any sector of one circle is congruent to any sector in another circle, then the two circles are congruent.

25. The bisector of the angle between a tangent and a chord bisects the intercepted arc.

BOOK III. TEST I

Completions (III—C-1)

1. If four numbers are in proportion, the first is to the third as the _____ is to the _____.

2. If a line is parallel to one side of a triangle, it divides the other two sides _____.

3. If the sides of two triangles are respectively proportional, the triangles are _____.

4. Two chords of a circle intersect. The segments of one are 3 and 5. The length of the other is 12. The segments of the latter are _____ and _____.

5. The sides of a right triangle are 5, 12, and 13. The segments into which the longest side is divided by the altitude upon it are _____ and _____.

6. A tangent and a secant are drawn from a point, the secant passing through the center of the circle. The length of the tangent is 10 in.; the external segment of the secant is 4 in. The radius is _____ in.

7. From a point E in a chord AB of a circle a perpendicular EC is drawn to the diameter AD. If AE is 5, EB is 2, and EC is 3, the radius is _____.

8. The hypotenuse of a right triangle is 10, and the altitude upon it is 5. The product of the arms is _____.

True-False Questions (III—TF-1)

1. Two dissimilar polygons may have all the angles of one equal respectively to all the angles of the other.

2. The median to the hypotenuse of a right triangle divides it into two similar triangles.

3. The perimeters of similar triangles are equal.

4. A diagonal divides a parallelogram into two similar triangles.

5. If the sum of the squares of two sides of a triangle equals the square of the third side, the triangle is a right triangle.

6. If two triangles have an altitude of one equal to an altitude of the other, they are congruent.

7. Mutually equiangular parallelograms are similar.

8. Two isosceles triangles are similar if a vertex angle of one equals a vertex angle of the other.

9. If a line cuts off from a triangle another triangle similar to the first, the line is parallel to the base.

10. If the perpendicular bisectors of the sides of a quadrilateral meet at a point, the product of the segments of one diagonal equals the product of the segments of the other.

11. A triangle similar to an isosceles triangle is isosceles.

12. A triangle whose sides are 7, 24, and 25 is a right triangle.

13. If a bisector of an angle of a triangle divides it into two similar triangles, the triangle is isosceles.

14. A line parallel to the bases of a trapezoid divides it into similar trapezoids.

15. The median to the hypotenuse of a right triangle divides it into two similar triangles.

16. If two similar polygons have a side of one equal to a side of the other, they are congruent.

17. If one of two similar polygons can be inscribed in a circle, the other can also.

18. If two polygons can be divided into the same number of triangles similar each to each, they are similar.

19. If C is a point in a tangent to a circle at the end A of a diameter AB, and CB cuts the circle at D, then CB times DB equals the square of the diameter.

20. If there exists a line from a vertex of a triangle to a point in the opposite side which cuts off a triangle similar to the original triangle, then the original triangle is a right triangle.

21. Two trapezoids are congruent if the sides of one are equal respectively to the sides of the other.

22. The squares of the arms of a right triangle have the same ratio as the adjacent segments into which the hypotenuse is divided by the altitude upon it.

23. If two triangles are similar and two sides of one are respectively parallel to two sides of the other, then the third sides are parallel.

24. Two triangles are equal if a pair of sides of one is equal to a pair of sides of the other and the sums of the angles opposite these sides are equal.

25. The only line from a vertex of a triangle to a point in the opposite side which divides that side into two segments proportional to the adjacent sides is the median.

TEST II

Completions (III—C-2)

1. If four numbers are in proportion, the second is to the first as the _____ is to the _____.

2. The bisector of an angle of a triangle divides the side opposite into segments proportional to _____.

3. If two triangles are mutually equiangular, they are _____.

4. The sides of a triangle are 4, 8, and 10. The greatest side of a similar triangle is 15. The segments of side 15 made by the bisector of the angle opposite are _____ and _____.

5. If two tangents to a circle of radius 5 in. are perpendicular to each other, their lengths are _____ in. and _____ in.

6. Two circles, of radii 5 in. and 8 in., are tangent externally. Chords are formed by a line through the point of contact. The chord of the larger circle is 6 in. The chord of the smaller is _____ in.

7. The altitude on the hypotenuse of a right triangle divides it into segments of 8 in. and 12 in. The shorter leg is _____ in.

8. Two circles, of radii 6 in. and 8 in., are tangent externally. The length of a common external tangent is _____ in.

True-False Questions (III—TF-2)

1. A mean proportional between two numbers equals the square of their product.

2. A triangle can always be constructed similar to a given one and having its perimeter equal to a given line segment.

3. If from a point outside a circle a secant terminating in the circle and a tangent be drawn, the square of the tangent equals the whole secant diminished by its external segment.

4. Lines joining the corresponding trisection points of two sides of a triangle are parallel to the third side.

5. If no two sides of two similar triangles are parallel, the corresponding angles are not equal.

6. If one of two similar polygons is regular, the other is regular also.

7. Two triangles are similar if their angles are proportional.

8. If the hypotenuse and an arm of one right triangle are proportional to the hypotenuse and an arm of another, the triangles are similar.

9. A line which is not parallel to a side of a triangle does not divide the other two sides proportionally.

10. A line parallel to a side of a parallelogram and cutting two sides divides it into two similar parallelograms.

11. Parallelograms whose sides are proportional are similar.

12. In an isosceles triangle it is possible to draw an altitude which will divide the triangle into two similar triangles.

13. A circle which passes through three vertices of a quadrilateral in which the product of the segments of one diagonal is equal to the product of the segments of the other will also pass through the fourth vertex.

14. Polygons similar to the same or similar figures are similar to each other.

15. Corresponding altitudes of similar triangles are equal.

16. The quadrilateral formed by joining the mid-points of the segments of the diagonals of a parallelogram is similar to the parallelogram.

17. If two parallelograms have their sides respectively parallel, they are similar.

18. If the three altitudes of one triangle are proportional to the three altitudes of another, the triangles are similar.

19. The square of the hypotenuse of a right triangle is equal to one half the sum of the square of the difference of the arms and the square of their sum.

20. If two angles of one triangle are equal respectively to two angles of another triangle, then the corresponding sides of the two triangles are in proportion.

21. If the perimeters of two polygons have the same ratio as any two sides, the polygons are similar.

22. Two polygons are similar if two angles of one equal two angles of the other and all the angles of both polygons are acute.

23. If three lines cut off corresponding proportional segments on two transversals, the three lines are parallel.

24. If the diagonals of a quadrilateral intersect each other so that the segments of one are proportional to the segments of the other, the quadrilateral is a trapezoid.

25. If the square of one side of a triangle exceeds the sum of the squares of the other two sides, the angle opposite this side is acute.

BOOK IV. TEST I

Completions (IV—C-1)

1. The area of a triangle is the product of its base and _____.

2. The areas of two similar convex polygons are to each other as the squares of _____.

3. The areas of two triangles with equal bases are to each other as _____.

4. If in a parallelogram having an area of 10 sq. in. lines join the mid-point of one side to the two opposite vertices of the parallelogram, the area of the triangle formed is _____ sq. in.

5. The corresponding sides of two similar polygons are 14 and 18. The area of the smaller polygon is 2500 sq. ft. The area of the larger polygon is _____ sq. ft.

6. The area of a rhombus whose diagonals are 12 and 24 is _____.

7. One of two similar polygons has a side of 5 in., while the corresponding side of the other is 12 in. Another polygon is similar to both of these and is equivalent to their sum. The corresponding side of the third polygon is _____ in.

8. From a point inside an equilateral triangle of side 5, perpendiculars are drawn to the three sides. The sum of the perpendiculars is _____.

True-False Questions (IV—TF-1)

1. Two rectangles are unequal if their bases and altitudes are unequal.

2. The area of a parallelogram equals the product of a diagonal and the perpendicular upon it from one of the vertices.

3. The areas of two similar triangles are to each other as their perimeters.

4. Isosceles triangles with equal bases and equal altitudes are congruent.

5. Triangles having equal bases are to each other as the medians to these bases.

6. If two unequal triangles have two sides of one equal to two sides of the other, the triangle with the smaller included angle is the greater.

7. Segments drawn from the mid-point of one diagonal of a quadrilateral to the other two vertices divide the quadrilateral into two equivalent quadrilaterals.

8. If two parallelograms have an angle of one equal to an angle of the other, their areas are to each other as the products of the sides including these angles.

9. A parallelogram is bisected by any line through the intersection of the diagonals.

10. The altitude on the hypotenuse of a right triangle divides it into two triangles whose areas have the same ratio as corresponding sides.

11. If two tangents to a circle intersect at right angles, the area of the triangle formed by lines connecting the common point and the two points of contact is one fourth the square of the radius.

12. The area of a right triangle equals one fourth the square of the sum of the arms minus the square of the radius of the circumscribed circle.

13. If the area of a parallelogram equals the product of two adjacent sides, it is a rectangle.

14. If the bisector of an angle of a triangle divides it into two equivalent triangles, the triangle is isosceles.

15. The altitudes of a triangle are inversely proportional to their corresponding bases.

16. Any line from a vertex of a triangle to a point in the opposite base divides it into two triangles whose areas are to each other as the segments of the base.

17. The sum of the squares of the sides of a polygon is to the sum of the squares of the sides of a similar polygon as the square of the sum of the sides of the first polygon is to the square of the sum of the sides of the second.

18. If an altitude of one equilateral triangle is double that of another, the area of the first triangle is double that of the second.

19. If diagonals of a parallelogram divide it into four equal triangles, the parallelogram is a square.

20. If one side of a triangle whose area is $\frac{a^2}{4}\sqrt{3}$ is a, the triangle is equilateral.

21. The areas of two triangles are to each other as their bases.

22. If a triangle and a parallelogram have the same area, the latter has the greater perimeter.

23. The sum of the perpendiculars upon the sides of a polygon from any point within is constant.

24. If the area of a polygon is equal to one half of the product of one side and the perpendicular upon it from any vertex, the polygon is a triangle.

25. If the areas of two triangles are to each other as the products of any pair of corresponding sides, the triangles are similar.

TEST II

Completions (IV—C-2)

1. The area of a parallelogram is the product of its base and _____.

2. The areas of two similar triangles are to each other as the squares of _____.

3. The areas of two triangles with equal altitudes are to each other as _____.

4. The mid-points of two sides of a triangle of area 10 sq. in. are joined to a point in the third. The area of the quadrilateral obtained is _____ sq. in.

5. A regular hexagon of side 2 has an area twice that of another regular hexagon. The perimeter of the latter is _____.

6. One angle of a rhombus of side 10 in. is 60°. The area of the rhombus is _____ sq. in.

7. The diagonals of parallelogram $ABCD$ intersect at O. E and F are the mid-points of BC and CD respectively. The area of the parallelogram is 48 sq. in. The area of triangle EFO is _____ sq. in.

8. The sum of the perimeters of two equilateral triangles is 45 in., and the area of one is 16 times that of the other. The perimeter of one is _____ in. and of the other is _____ in.

True-False Questions (IV—TF-2)

1. The area of a rhombus equals the product of its diagonals.

2. The areas of two similar polygons are to each other as the areas of any two of the corresponding similar triangles into which the polygons can be divided.

3. The line joining the mid-points of two sides of a triangle cuts off one half of the triangle.

4. The area of a parallelogram equals the product of two adjacent sides.

5. A rhombus and a square with equal sides have unequal areas.

6. If the mid-points of the sides of a quadrilateral are joined to form another quadrilateral, the area of the smaller is one half that of the greater.

7. If the areas of two triangles are to each other as the products of two sides, the included angles are equal.

8. The only line from a vertex of a triangle to a point in the opposite side which divides it into two equivalent triangles is the median.

9. If two right triangles have the same area, they are similar.

10. All triangles with the same base and equal areas have their vertices in one line parallel to the base.

11. If two similar polygons have the same area, they are congruent.

12. If a point inside a parallelogram is joined to the four vertices, the sum of two triangles whose bases are parallel is one half of the parallelogram.

13. The areas of two triangles are to each other as the products of their three sides.

14. The areas of two right triangles inscribed in the same circle are to each other as the altitudes on the hypotenuse.

15. A quadrilateral can always be formed equivalent to a given polygon.

16. If the area of a parallelogram is one half the product of its diagonals, it is a square.

17. Of all parallelograms having a given base and a given area, the rectangle has the least perimeter.

18. The area of an isosceles right triangle is one half the square of the hypotenuse.

19. If two triangles have two sides of one equal to two sides of the other and the third side of the first double the third side of the second, then the area of the first is double that of the second.

20. Two parallelograms are unequal if their bases and altitudes are unequal.

21. Two noncongruent rectangles with equal perimeters cannot have the same area.

22. Two isosceles triangles having the altitude of one equal to half the base of the other and the sides opposite the base angles all equal are equal.

23. Squares inscribed in the same or equal circles are equal.

24. If two quadrilaterals have areas proportional to the squares of lines drawn one in each figure, then the two quadrilaterals are similar.

25. If the area of a quadrilateral is one half the product of the diagonals, it is a rhombus or a square.

BOOK V. TEST I

Completions (V—C-1)

1. If a circle is divided into three or more equal arcs, the chords joining the adjacent points of division form a _____.

2. The circumference of a circle of radius R is _____.

3. If each of two regular polygons has the same number of sides, the polygons are _____.

4. If a circle of radius 5 has an area four times that of another, the radius of the second circle is _____.

5. The area of a sector of $B°$ in a circle of radius 10 is _____ times the area of one of $B°$ in a circle of radius 2.

6. If the circumference and area of a circle are numerically equal, the radius is _____.

7. If two circles of radii 6 and 12 are concentric, the portion of the larger circle cut off by a tangent to the smaller is _____.

8. A regular hexagon circumscribed about a circle of radius 5 has an area of _____.

True-False Questions (V—TF-1)

1. The apothem of a regular polygon is the radius of its circumscribed circle.

2. A regular polygon can be inscribed in a circle.

3. If a circle can be inscribed in a polygon, the polygon is regular.

4. A segment of a circle is the portion cut off by two radii.

5. The area of a sector of a circle is the product of the radius and the arc of the sector.

6. An equilateral polygon circumscribed about a circle is regular if the number of sides is even.

7. In different circles the areas of sectors whose angles are equal are proportional to the squares of the radii.

8. The area of a regular inscribed hexagon is three fourths that of the regular circumscribed hexagon.

9. The perimeter of a regular hexagon is three times the diameter of the circumscribed circle.

10. Of all regular polygons which can be inscribed in a circle, the octagon has the greatest area.

11. The areas of two circles are to each other as their circumferences.

12. The area of a sector is to the area of the circle as the arc of the sector is to the circumference.

13. A circle can be circumscribed about any regular polygon with an even number of sides.

14. A circle can be divided into any number of equal parts with the ruler and compasses.

15. The bisector of an angle of a triangle divides the opposite side into extreme and mean ratio.

16. A line is divided in extreme and mean ratio if the square of the large segment equals the product of the line and the short segment.

17. The area of a ring included between two concentric circles is the area of a circle whose diameter is a chord of the larger tangent to the smaller.

18. If the radius is doubled, the area of the circle is doubled.

19. The areas of two similar triangles are to each other as the areas of their circumscribed circles.

20. If the circumference of a plane figure is π times the distance across it, the figure is a circle.

21. If the perimeter of a regular circumscribed polygon is four times that of the regular inscribed polygon of the same number of sides, the polygon is a triangle.

22. The area of a polygon and the area of a circle are to each other as the perimeter of the polygon is to the circumference of the circle.

23. Regular polygons of the same number of sides circumscribed about the same circle are congruent.

24. A plane cannot be filled with regular pentagons.

25. If the perpendicular bisectors of the sides and the bisectors of the angles of a polygon all meet at a common point, the polygon is regular.

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